Control charting normal variance – reflections, curiosities, and recommendations

Sven Knoth

September 2007

Outline

1. Introduction
2. Modelling
3. Two-sided EWMA charts for variance
4. Conclusions

Introduction

Aim of control charting is to detect deviations from stability

- as fast as possible
- without too many false alarms.

Parameters characterizing stability are

- mean level,
- scale (uniformity, variance, repeatability),
- ...

Why variance?

- Ensure appropriate control limits for mean chart.
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1. CD (critical dimension) uniformity:
   - Measure a certain number (20 ... 200) of, e. g., lines of nominal size 200 *nm* on a single plate,
   - calculate sample mean $\bar{CD}$ and standard deviation S_{CD},
   - chart both.

2. Gauge repeatibility – CD-SEM (scanning electron microscope):
   - Repeat a few times (e. g., 5) the measurement of one given line,
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Some remarks about variance monitoring

- Variance components: Yashchin (1994), Woodall & Thomas (1995), Srivastava (1997),
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- small or large batch sizes,

Focus: Small batch sizes larger 1, one variance component only.

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Modelling

Sequence $\{X_{ij}\}$, $i = 1, 2, \ldots$ and $j = 1, 2, \ldots, n > 1$

with $X_{ij} \sim \mathcal{N}(\mu, \sigma^2)$, independence.

The change-point model: For a certain unknown m

$$\sigma^2 = \begin{cases} \sigma_0^2 = 1 & , \; i < m \\ \sigma_1^2 \neq \sigma_0^2 & , \; i \geq m \end{cases} .$$

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Pre-processing of batch data

In order to monitor σ the usual suspects are

$$R_i = \max_j X_{ij} - \min_j X_{ij} \,,$$

$$S_i^2 = \frac{1}{n-1}\sum_{j=1}^{n}\left(X_{ij} - \bar{X}_i\right)^2 \quad , \; \bar{X}_i = \frac{1}{n}\sum_{j=1}^{n} X_{ij} \,,$$

$$S_i = \sqrt{S_i^2} \,,$$

$$lS_i^2 = \log S_i^2 \,,$$

$$abcS_i^2 = a + b \, \log(S_i^2 + c) \,.$$

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Why log?

1. Box, Hunter & Hunter (1978) recommended it.
2. It transforms scale-change into level change.
3. The variance of $\log S^2$ does not depend on σ.
4. New statistic behaves nearly "normally" and
5. is, of course, more symmetric.

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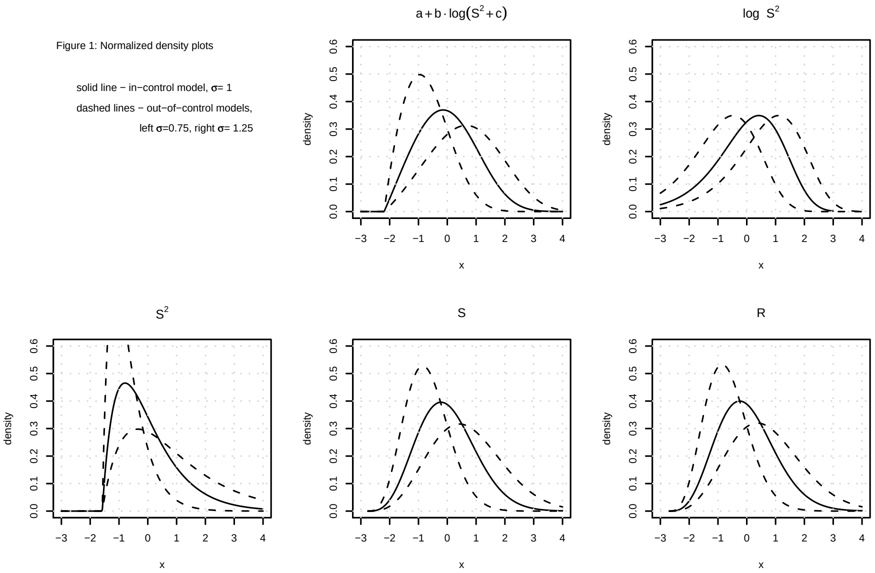

Figure 1: Normalized density plots

solid line − in−control model, σ= 1

dashed lines − out−of−control models,

left σ=0.75, right σ= 1.25

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Who is who in $\log S^2$-SPC

- Crowder & Hamilton (1992), EWMA,
- Chang & Gan (1994), EWMA,
- Chang & Gan (1995), CUSUM,
- Amin & Wilde (2000), Crosier-type CUSUM,
- Castagliola (2005), $a + b \log(S^2 + c)$ EWMA,
- ...

Short list of comparison papers

- Tuprah & Ncube (1987),
- Srivastava & Chow (1992),
- Lowry, Champ & Woodall (1995),
- Mittag, Stemann & Tewes (1998),
- Acosta-Mejía, Pignatiello Jr. & Rao (1999),
- Poetrodjojo, Abdollahian & Debnath (2002),
- ...

Further transformations

- HAWKINS (1981), $\frac{|(X-\mu_0)/\sigma_0|^{1/2}-.82218}{.34914}$,
- APR (1999), $\Phi^{-1}\left[F_{\chi^2_{n-1}}\left(\frac{(n-1)S^2}{\sigma_0^2}\right)\right]$,
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Two-sided EWMA charts for variance

$$V_i \in \left\{ S_i^2, S_i, R_i, \log S_i^2, a + b \, \log(S_i^2 + c) \right\},$$

$$Z_0 = z_0 = E_\infty(V_i)\,,$$

$$Z_i = (1-\lambda)\, Z_{i-1} + \lambda\, V_i \quad , \; i \geq 1\,,$$

$$L = \min \left\{ i \in \mathbb{N} : Z_i \notin [c_l, c_u] \right\}.$$

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Comparison study

1. Calibrate all schemes to give $E_\infty(L) = 500$.
2. Deploy "ARL-unbiased" designs (see APR (1999)).
3. Look for "optimal" λ, that is, minimize

   $$\mathcal{L}_{0.75} + \mathcal{L}_{1.25} \quad \text{and} \quad \mathcal{L}_{0.5} + \mathcal{L}_{1.5} \quad \text{, respectively,}$$

   among $\lambda \in \{0.02, 0.03, \ldots, 0.99, 1.00\}$.
4. Optimal values for λ are:

case	statistic				
	R	S^2	S	IS^2	$abcS^2$
$\mathcal{L}_{0.75} + \mathcal{L}_{1.25}$	0.08	0.08	0.08	0.07	0.08
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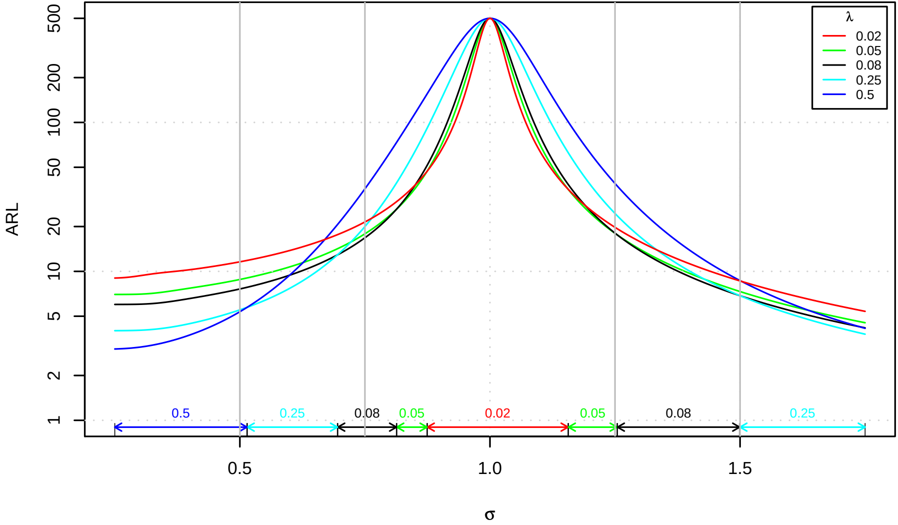
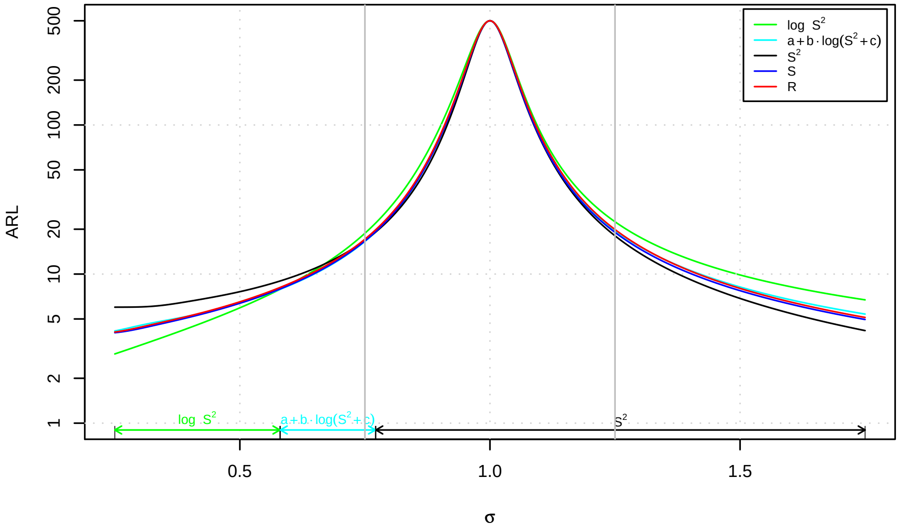
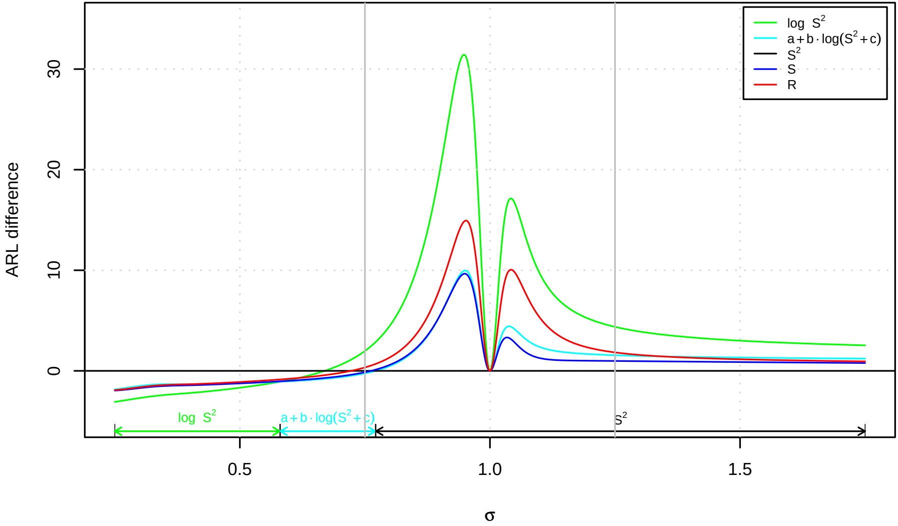
Illustration for S^2 EWMA



Competition for minimal $\mathcal{L}_{0.75} + \mathcal{L}_{1.25}$



Competition for minimal $\mathcal{L}_{0.75} + \mathcal{L}_{1.25}$ II



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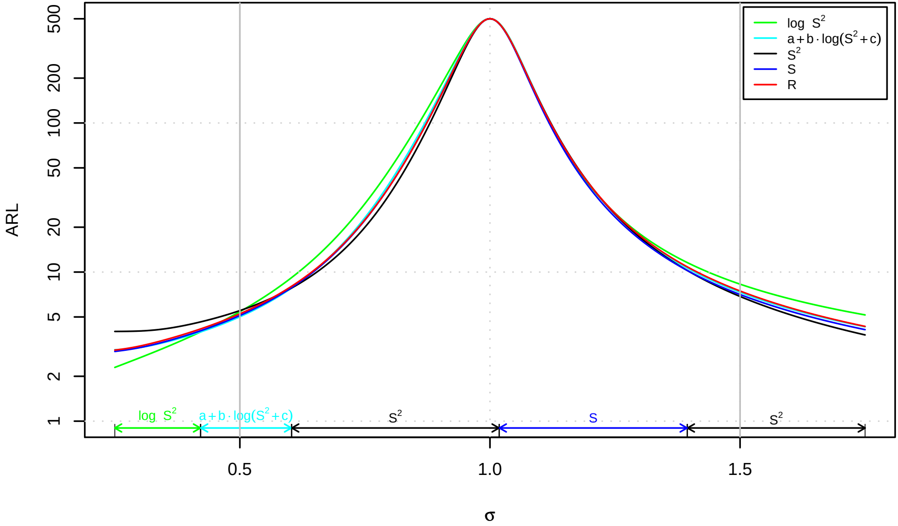
Competition for minimal $\mathcal{L}_{0.75} + \mathcal{L}_{1.25}$ III

σ	statistic				
	IS^2	$abcS^2$	S^2	S	R
0.4	**4.374**	5.251	6.575	*5.143*	5.249
0.5	**5.939**	6.389	7.619	*6.374*	6.514
0.6	8.547	**8.409**	9.438	*8.459*	8.660
0.7	13.74	**12.59**	13.17	*12.63*	12.96
0.75	18.78	**16.56**	16.81	*16.67*	17.14
0.8	27.94	*23.92*	**23.44**	24.04	24.78
0.9	96.70	*82.24*	**76.74**	82.26	84.96
1.0			500.000		
1.1	90.80	83.53	**81.16**	*82.43*	86.43
1.2	30.74	27.27	**25.61**	*26.61*	27.88
1.25	22.44	19.61	**18.06**	*19.04*	19.89
1.3	17.67	15.26	**13.77**	*14.73*	15.35
1.4	12.54	10.60	**9.206**	*10.12*	10.51
1.5	9.866	8.190	**6.864**	*7.740*	8.017
1.6	8.235	6.735	**5.460**	*6.295*	6.509

Competition for minimal $\mathcal{L}_{0.5} + \mathcal{L}_{1.5}$



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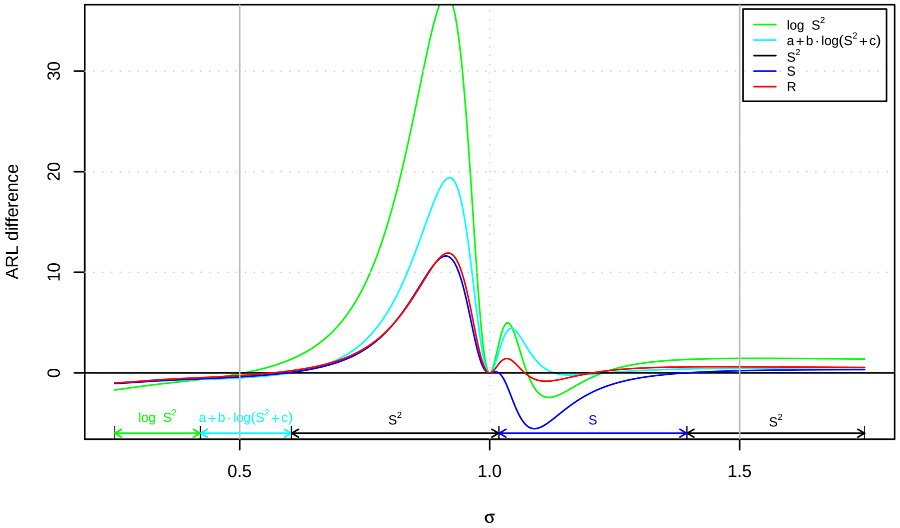

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Competition for minimal $\mathcal{L}_{0.5} + \mathcal{L}_{1.5}$ III

σ	statistic				
	IS^2	$abcS^2$	S^2	S	R
0.3	**2.662**	3.120	4.025	*3.114*	3.190
0.4	**3.677**	*3.763*	4.460	3.824	3.935
0.5	5.425	**5.014**	5.516	*5.113*	5.249
0.6	8.997	**7.678**	*7.703*	7.736	7.896
0.7	18.22	14.79	**13.35**	*14.47*	14.63
0.8	49.01	39.91	**33.51**	*37.94*	37.96
0.9	174.1	155.7	**137.4**	*148.8*	148.9
1.0			500.000		
1.1	*136.3*	139.3	138.4	**133.0**	137.7
1.2	*37.89*	38.31	38.44	**36.39**	38.39
1.3	17.99	17.30	*17.07*	**16.57**	17.54
1.4	11.34	10.40	**10.00**	*10.02*	10.59
1.5	8.299	7.327	**6.862**	*7.072*	7.465
1.6	6.611	5.674	**5.180**	*5.466*	5.760
1.7	5.552	4.664	**4.158**	*4.474*	4.707

Conclusions

1. None of the statistics provides ARL performance that is symmetric in σ.
2. The $\log S^2$ seems to be the worst approach, even beaten by R.
3. The newer $a + b\log(S^2 + c)$ is considerably better than $\log S^2$. But, these efforts do not really pay off.
4. There is no reason to deploy log based approaches at all. This is supported also by one-sided results (both EWMA and CUSUM).
5. For application, one should prefer S^2 and S. The latter is the most popular quantity at AMTC.

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Backup

Some upper variance charts

Slightly modified and shortened update of Table 5 in Chang & Gan (1995) – the EWMA schemes are also one-sided and equipped with a lower reflecting barrier, see Knoth (2005) for more details.

	CUSUM-S^2	CUSUM-ln S^2	EWMA-S^2	EWMA-ln S^2
	$k_h = 1.285$	$k_h^{\ln} = 0.309$	$\lambda = 0.15$	$\lambda^{\ln} = 0.28$
σ	$h_h = 2.922$	$h_h^{\ln} = 1.210$	$c = 2.4831$	$c^{\ln} = 1.4085$
1	100.0	100.0	100.0	100.0
1.1	27.9	30.2	27.9	30.0
1.2	12.8	13.8	12.9	13.8
1.3	**7.75**	**8.15**	**7.86**	**8.26**
1.4	5.47	5.63	5.57	5.76
1.5	4.22	4.29	4.30	4.43
2	2.08	2.11	2.11	2.22

Numerical handling of the sample range R

Bland, Gilber, Kapadia & Owen (1966):

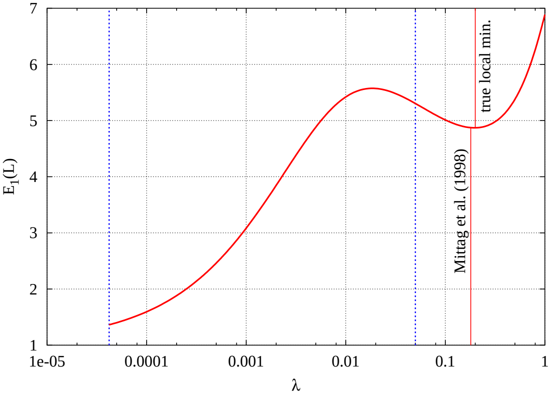
$$P(R/\sigma \leq r) = \int_{\infty}^{\infty} n\,\phi(x)\big(\Phi(x+r) - \Phi(x)\big)^{n-1}\,dx\,.$$

ARL integral equations and it's solution

$$\mathcal{L}(z) = 1 + \int_{c_l}^{c_u} \mathcal{L}(x) \, \frac{1}{\lambda} \, f\left(\frac{x - (1-\lambda)\, z}{\lambda}\right) \, dx \quad , \; z \in [c_l, c_u] \, .$$

1. $\log S^2$: Gauss-Legendre Nyström,
2. others: collocation with piece-wise Chebyshev polynomials,
3. validated with Monte Carlo with 10^8 replicates.

The λ hunt for minimal out-of-control ARL



$\lambda = 0.000\,042$,

$c = 0.000\,064\,375\,308$,

$\widehat{E_\infty(L)} = 250.103 \pm 0.091$,

$\widehat{E_1(L)} = 1.3628 \pm 0.0000$,

10^9 rep.

$P_\infty(L = 1) \approx 0.4!$