EWMA control charts for monitoring normal variance

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Outline

1. Variance monitoring,
2. EWMA,
3. Numerical challenges,
4. Application,
5. Conclusions.

Statistics & Monitoring

Statistical Process Control (SPC)	industrial statistics
Change Point Detection Schemes	mathematical statistics
Surveillance	biostatistics
Structural Breaks	econometrics

Objective of "Control Charting"

WOODALL/MONTGOMERY (1999)

... in the area of control charting and SPC. As a general definition, we include in this area any statistical method designed to detect changes in a process over time.

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- more formal objective of SPC:

  detect m as fast and reliable as possible.

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- single variance change or simultaneous mean/variance change.

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$$L = \inf\left\{i \in \mathbb{N} : Z_i > \sigma_0^2 + c\,\sigma_Z\right\}.$$

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use S_i or $\ln S_i^2$ or R_i or $\ldots$

Performance measures for control charting

Denote $E_m(\cdot)$ the expectation for a change point at m.

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3. Roberts' steady state: $E_\infty(L)$ & $\lim_{m\to\infty} E_m\big(L-m+1|L \geq m-1\big)$
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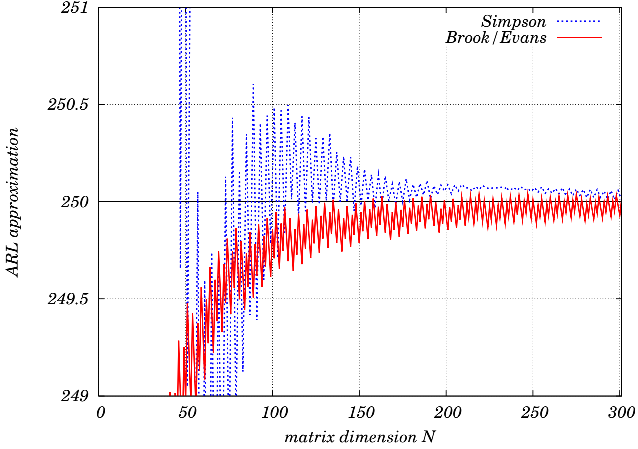
6. Collocation (Fellner, 1990, CUSUM)
   Gianino/Champ/Rigdon (1990).

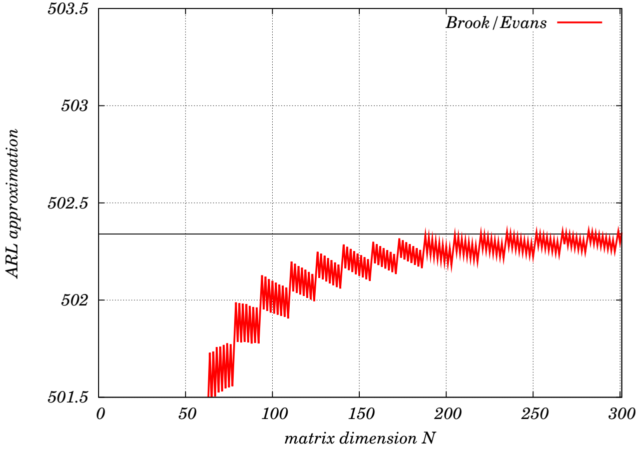
Application to upper S^2 EWMA ARLs I

$n = 5,\ \lambda = 0.18,\ c = 2.90922$ (final $E_\infty(L)$ value is 250)



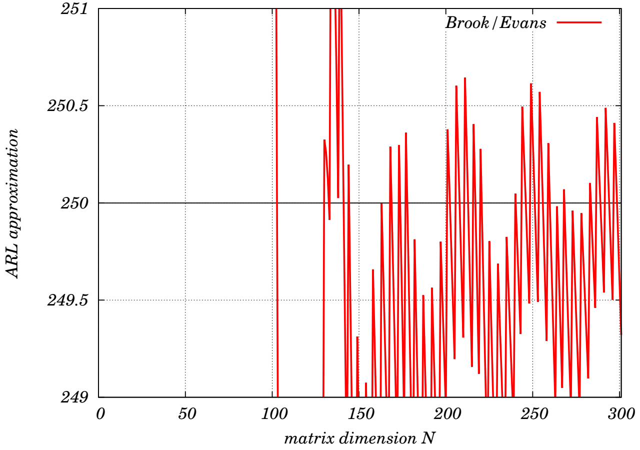
Application to upper S^2 EWMA ARLs II

$n = 3$, $\lambda = 0.18$, $c = 3.61384$ (final $E_\infty(L)$ value is 502.34)



Application to upper S^2 EWMA ARLs III

$n = 2,\ \lambda = 0.025,\ c = 1.66186$ (final $E_\infty(L)$ value is 250)



Numerics

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- $f(z_{i-1} \to z_i) = 0$ for $z_i < (1-\lambda) z_{i-1} \rightsquigarrow$ Nyström fails,
- product Nyström,
- collocation: $ARL(z) = \sum_{i=1}^{N} c_i T_i(z)$,

  $T_i(z)$ – monomials, Lagrange or Chebyshev polynomials

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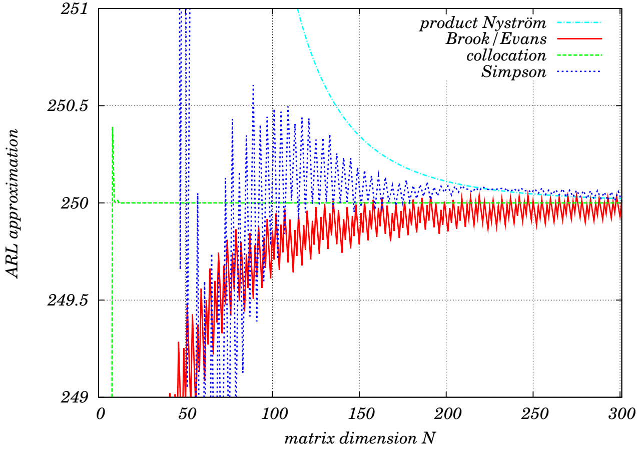

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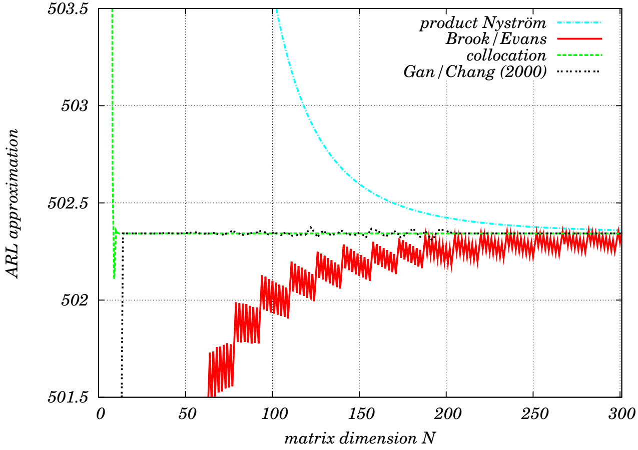

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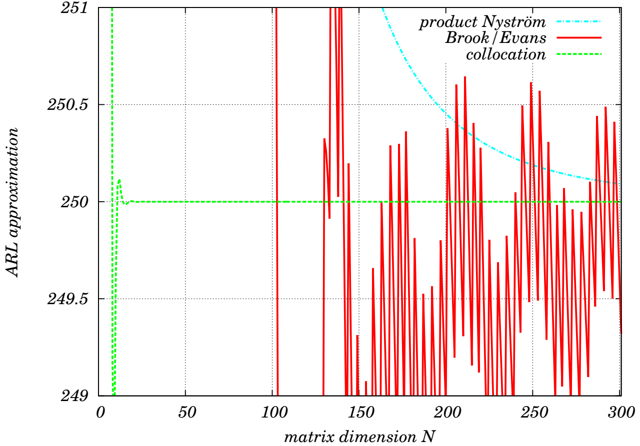

How fast is the collocation approach?

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Method	matrix dimension N				
	25	51	101	201	301
Markov chain	103.7077	307.4809	254.6729	250.3782	249.3206
	< 1[1]	1	3	13	39
factor Nyström	< 0	376.0594	256.6795	250.4456	250.0908
	< 1	1	2	9	27
Collocation	249.9999	249.9997	249.9997	249.9997	249.9997
	1	3	11	40	101

[1] CPU time in hundredth seconds on an Athlon XP 1.4 GHz

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- similar ideas could be employed for $\bar{X}$-S^2 EWMA charts – enhance Gan (1995),
- for $\ln S^2$ EWMA charts the Gauss-Legendre Nyström method is the most powerful – Crowder/Hamilton (1992).

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Setup: derived from Crowder/Hamilton (1992)

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3. both: $\lambda \in \{0.05, 0.16, 0.32\}$, $E_\infty(L) = 200$, $\sigma_0^2 = 1$.

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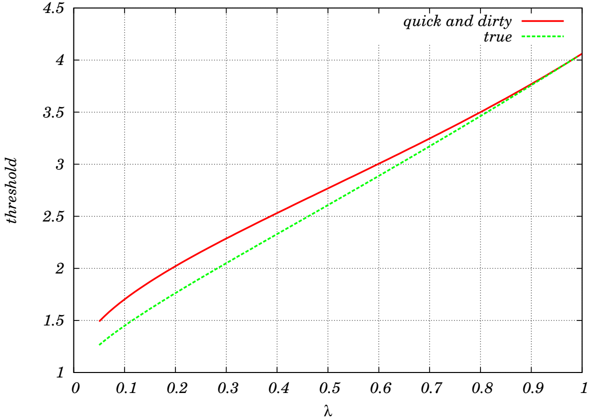
Results: ARL

σ^2	$\lambda = 0.05$			$\lambda = 0.16$			$\lambda = 0.32$		
	$\ln S^2$		S^2	$\ln S^2$		S^2	$\ln S^2$		S^2
	0	-.267		0	-.267		0	-.267	
1	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00
1.1^2	**43.04**	41.55	37.78	45.59	43.19	43.44	48.93	46.90	50.50
1.2^2	**18.10**	19.92	16.71	18.54	18.46	17.42	19.63	19.12	20.05
1.3^2	10.75	13.11	10.32	**10.52**	11.11	9.85	10.73	10.79	10.74
1.4^2	7.63	9.93	7.39	7.20	7.96	6.68	**7.08**	7.34	6.93
1.5^2	5.97	8.11	5.74	5.49	6.27	5.03	5.24	5.57	5.03
2^2	3.17	4.67	2.74	2.77	3.37	2.33	2.44	2.78	2.18

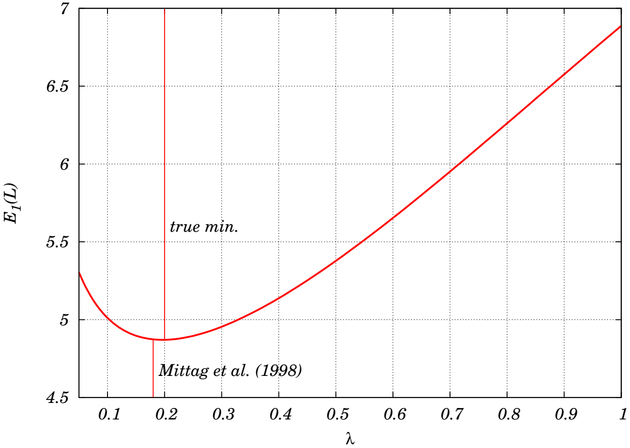
Calculating limits: quick and dirty vs. costly and correct

$$n = 5,\ E_\infty(L) = 370\,,\ \textit{threshold} = \sigma_0^2 + \sigma_0^2\sqrt{\frac{\lambda}{2-\lambda}} \times \begin{cases} c_{\text{coll}}\sqrt{\frac{2}{n-1}} & \text{, correct} \\ \frac{\chi^2_{n-1;1-1/370}}{n-1} - 1 & \text{, quick and dirty} \end{cases}$$



Running for the "optimal" λ

$n = 5$, $E_\infty(L) = 250$, $\sigma_1 = 1.5$, Mittag et al. (1998)



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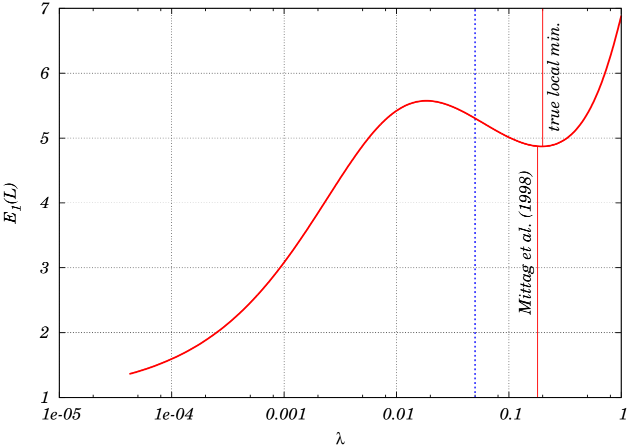

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$$P_\infty(L=1) \approx 0.4\,.$$

Conclusions

- EWMA control charts for the variance were often applied, but the accuracy in computing their ARL values was poor.
- Collocation allows to cure these accuracy problems.
- The algorithm will be implemented in R library spc.
- The quick and dirty approach of deriving S^2 EWMA control limits provides suitable results.
- Now, we are able to compute the ARL of S^2 EWMA control charts for very small λ.
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