Fast initial response features for EWMA Control Charts

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Usual EWMA scheme for the mean
(with fixed control limits)

$$X_n \sim \mathcal{N}(\mu_n, \sigma^2 = 1) \ , \quad \mu_n = \begin{cases} 0 & , \ n < m \\ \delta \ (= 1) & , \ n \geq m \end{cases} \ , \quad n = 1, 2, \ldots$$

$$Z_0 = z_0 = 0 \ , \quad Z_n \underset{n \geq 1}{=} (1 - \lambda) \, Z_{n-1} + \lambda \, X_n \ , \quad \lambda \in (0, 1] \, ,$$

$$E_m(Z_n) \underset{n<m}{=} 0 \quad \Big(E_\infty(Z_n) = 0 \text{ for all } n, \text{ for } z_0 \neq 0 \text{ at least } \lim_{n \to \infty} E_\infty(Z_n) = 0 \Big) \, ,$$

$$Var_m(Z_n) = \frac{2}{2 - \lambda} \left(1 - [1 - \lambda]^{2\,n} \right) \to \frac{2}{2 - \lambda} \text{ for } n \to \infty \, ,$$

$$L = \inf \left\{ n \in \mathbb{N} : |Z_n| > c \sqrt{\frac{2}{2 - \lambda}} \right\} \, .$$

Fast Initial Response (FIR)

Aim: Improve control chart performance at scheme start-up!

e. g.,

- following a process adjustment in quality control,
- after re-estimating the underlined model in monitoring financial data,
- one suspects a change before monitoring is started,
- weaken the influence of the initial setup (z_0) of the usual EWMA scheme,
- ...

Fast Initial Response – revisiting literature

- For the first time considered by Lucas/Crosier (1982) for CUSUM.
- Lucas/Saccucci (1990 ... 1988) `Markov chain`

  5.1 FIR Feature ... A disadvantage of this approach is that it requires <u>two</u> separate EWMA's ... A detailed discussion of this approach will be given in a future article.

  B.1 FIR Feature ... The FIR feature requires the simultaneous implementation of <u>two</u> one-sided EWMA ... We plan to implement the exact method for calculating FIR ARL's in our future work.

- Rhoads/Montgomery/Mastrangelo (1996), `Monte-Carlo`

  Fast initial response scheme for exponentially weighted moving average control chart.

- Steiner (1999), `Markov chain`

  EWMA control charts with time-varying control limits and fast initial response.

- Montgomery (2001): $\frac{1}{2}$ page in *Introduction to Statistical Quality Control*, 4^{th} *ed.*

FIR as remedy for inertia

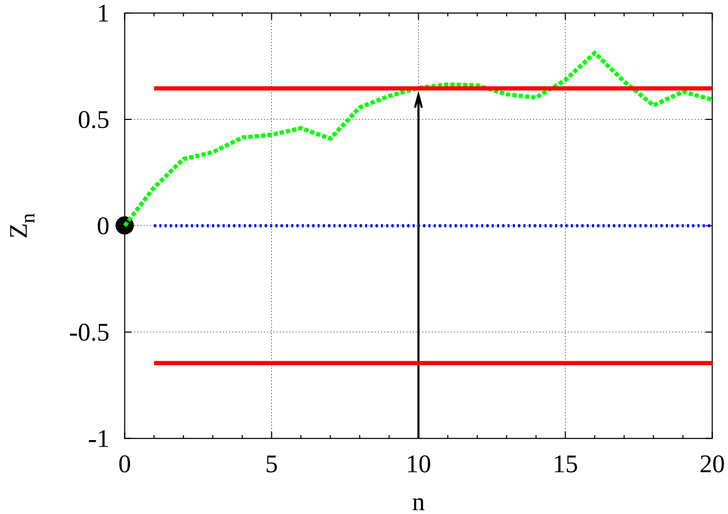
Problem: inert behavior of the usual EWMA scheme for early changes (small m), i. e. strong influence of the starting value z_0 at early time points if λ is small

$$Var_m(Z_n) = \frac{2}{2-\lambda}\left(1 - [1-\lambda]^{2n}\right) \ll \frac{2}{2-\lambda} = \lim_{n\to\infty} Var_m(Z_n)$$

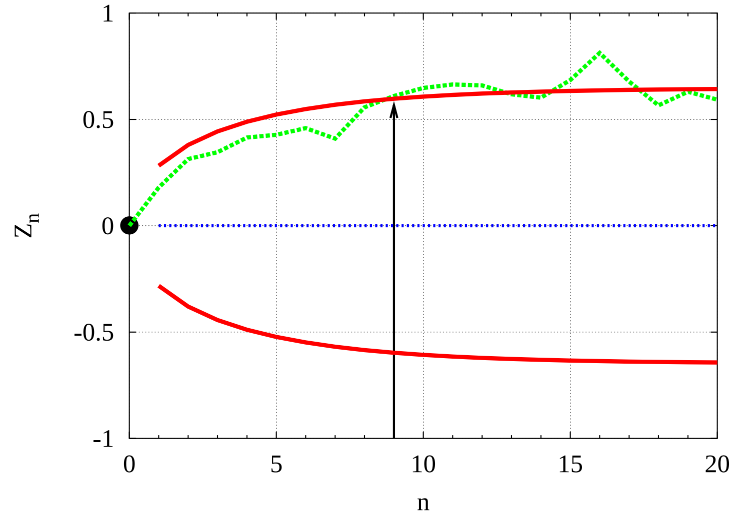
Motivating examples: $\lambda = 0.1$, $m = 1$, $E_\infty(L) = 500$

FIR EWMA examples I – $m = 1$

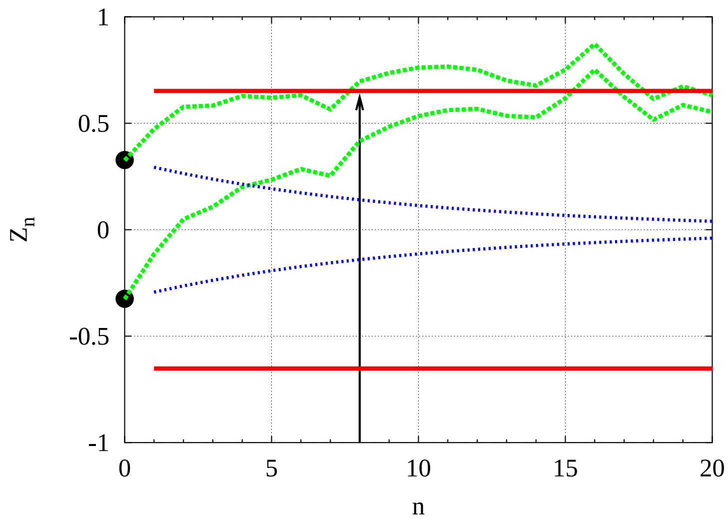
Roberts (1959), Wortham (1972), Ho (1978)



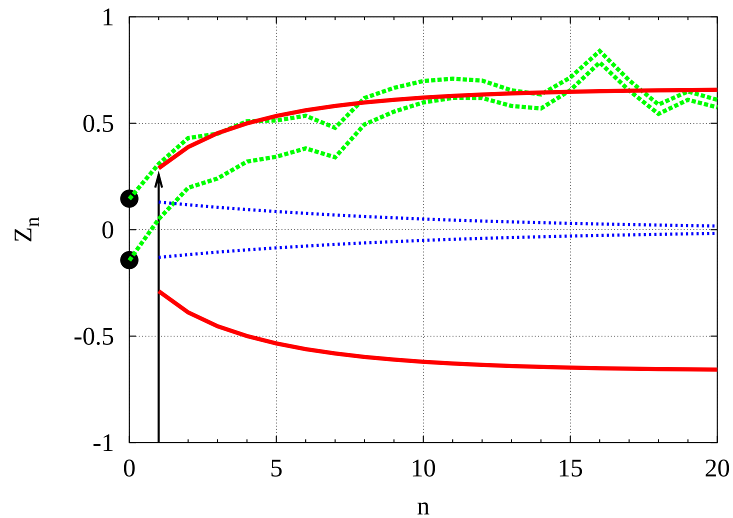
Roberts (1959), Chandrasekaran et al. (1995)



Lucas and Saccucci (1990)

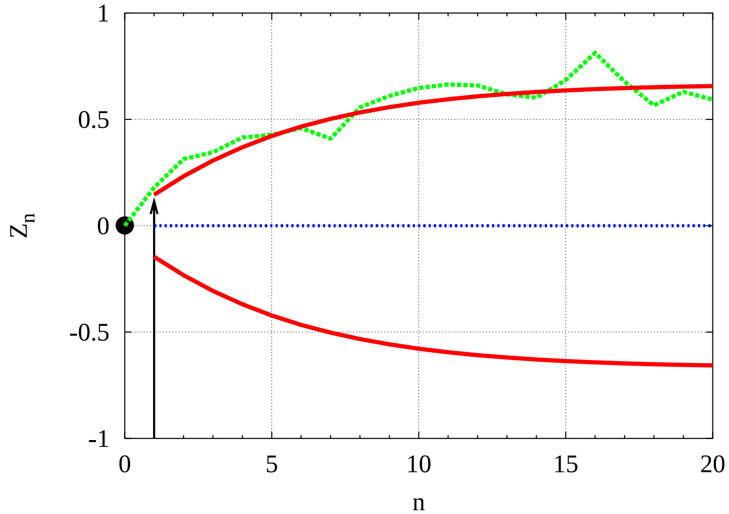


Rhoads et al. (1996)

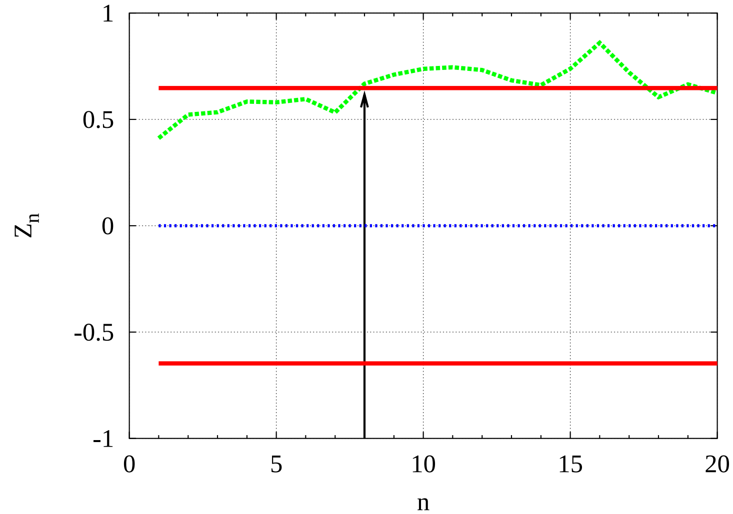


FIR EWMA examples II – $m = 1$

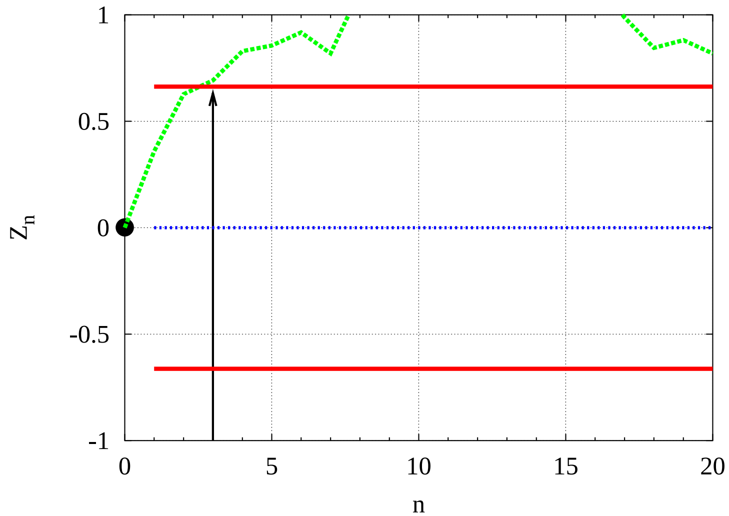
Steiner (1999)



(Knoth 2003)



Fink (1994)



?

?

Formulas

$$Z_0 = 0 \ , \ Z_n \underset{n\geq 1}{=} (1-\lambda)\, Z_{n-1} + \lambda\, X_n \ , \ c^* = c\sqrt{\frac{2}{2-\lambda}} \ , \ l_n = \sqrt{1-(1-\lambda)^{2n}}$$

raise alarm, if

(fcl) $|Z_n| > c^* \,,$

(vacl) $|Z_n| > c^*\, l_n \,,$

(fir) $\max\left\{|Z_n^u|, |Z_n^l|\right\} > c^* \ , \ Z_0^u = -Z_0^l = \frac{c^*}{2} \,,$ Lucas/Saccucci 1990

(fvacl) $\max\left\{|Z_n^u|, |Z_n^l|\right\} > c^*\, l_n \ , \ Z_0^u = -Z_0^l = l_1 \cdot \frac{c^*}{2} \,,$ Rhoads et al. 1996

(fadj) $|Z_n| > c^*\, l_n \left(1-(1-f)^{1+a(n-1)}\right) \ , \ f = 0.5, \ a = 0.3 \,,$ Steiner 1999

(stat) $|Z_n| > c^* \ , \ Z_1 = X_1\sqrt{2/(2-\lambda)} \,, \ Var_m(Z_n) = \frac{2}{2-\lambda} \,,$ (Knoth 2003)

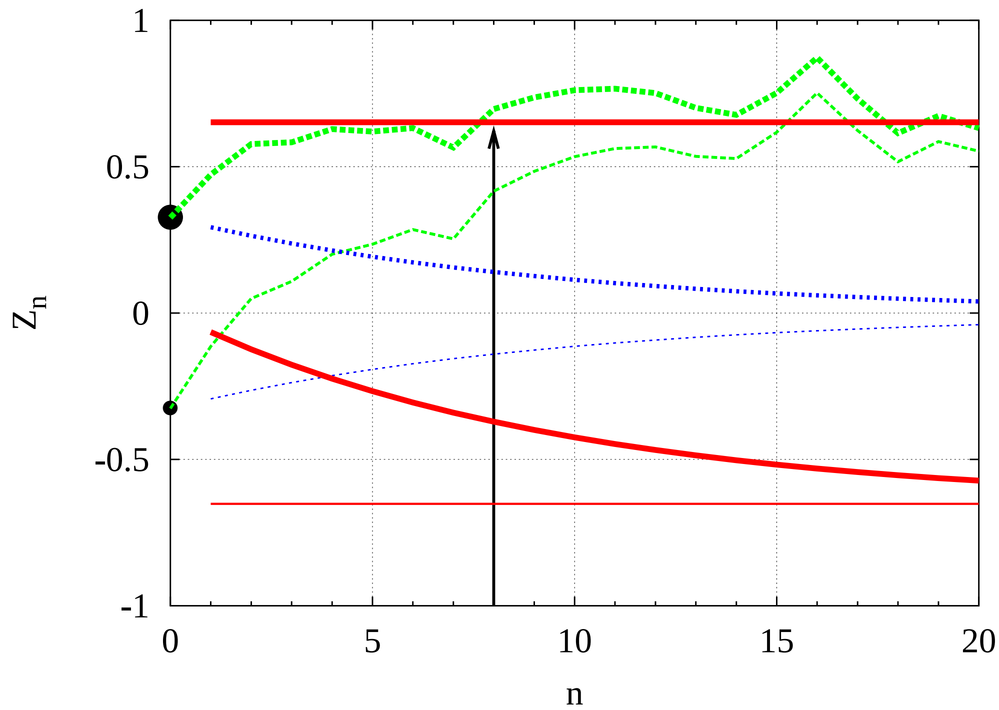
(switch) $|Z_n| > c^* \ , \ \lambda = \begin{cases} \lambda_0 & , \ n \leq n_1 \\ \lambda_1 < \lambda_0 & , \ n > n_1 \end{cases} \ , \ n_1 = 10, \lambda_1 = \lambda_0/2 \,.$ Fink 1994

FIR due to Lucas/Saccucci needs two charts?!

$$Z_0^u = -Z_0^l = \frac{c^*}{2}\,,$$

$$Z_n^l = Z_n^u - \left(Z_0^u - Z_0^l\right)(1-\lambda)^n$$

$$= Z_n^u - c^*\,(1-\lambda)^n$$



Therefore, $\left(L = \inf\left\{n \in \mathbb{N} : \ldots\right\}\right)$

$$\max\left\{|Z_n^u|, |Z_n^l|\right\} > c^* \Leftrightarrow \max\left\{Z_n^u, -Z_n^l\right\} > c^*$$

$$\Leftrightarrow Z_n^u \notin \left[-\left(1-(1-\lambda)^n\right)c^*, c^*\right]$$

Remark: Similar equivalence is valid for FIR due to Rhoads et al.

Computation of $E_m(L)$, critical value c, and ...

- For fixed limits (homogeneous) schemes (`fcl, stat`) a lot of methods available, for `switch` easy adaption possible, for `fir` at least the two-dimensional approach could be exploited (Lucas/Saccucci).

- For EWMA schemes with non-constant limits or varying transition probabilities literature offers:
  - modified Markov chain approaches, e. g., in Chandrasekaran et al. (1995) and Steiner (1999),
  - Monte-Carlo, e. g., in Rhoads et al. (1996),
  - density recursion method in Margavio, Conerly, Woodall, Drake (1995)

  false alarm rate $r_n = P_\infty(L = n | L \geq n)$ for `fcl, vacl` and a constant false alarm rate EWMA scheme

Density recursion I

A more general notation is $L = \inf \left\{ n \in \mathbb{N} : Z_n \notin [a_n, b_n] \right\}$.

Consider the following recursion:

$$f_1(z; z_0) = \begin{cases} \frac{1}{\lambda}\,\phi_\mu\!\left(\frac{z-(1-\lambda)z_0}{\lambda}\right) & , \; a_1 \le z \le b_1 \\ 0 & , \text{otherwise} \end{cases} ,$$

$$f_n(z; z_0) = \begin{cases} \int\limits_{a_{n-1}}^{b_{n-1}} f_{n-1}(\tilde{z}; z_0)\, \frac{1}{\lambda}\,\phi_\mu\!\left(\frac{z-(1-\lambda)\tilde{z}}{\lambda}\right) d\tilde{z} & , \; a_n \le z \le b_n \\ 0 & , \text{otherwise} \end{cases} .$$

Then $f_n^*(z; z_0) = f_n(z; z_0) / \int_{a_n}^{b_n} f_n(\tilde{z}; z_0)\, d\tilde{z}$

is a density of the EWMA statistic Z_n conditioned on $L > n$ and $Z_0 = z_0$,

where the denominator is equal to $P(L > n)$.

Density recursion II

Based on $P(L > n)$ one can obtain

$$\texttt{zero-state ARL} \qquad E_{1(\infty)}(L) = \sum_{n=0}^{\infty} P_{1(\infty)}(L > n)\,,$$

$$D_m = E_m(L - m + 1 | L \geq m) = \frac{\sum_{n=m}^{\infty} P_m(L > n)}{P_m(L > m - 1)}\,,$$

$$\texttt{steady-state ARL} \quad D = \lim_{m \to \infty} D_m \approx D_{m_0}\,,\; m_0 = 100\,,$$

$$P_m(L = n)\,,\; P_m(L = n + m | L \geq m)\,,\; L_\alpha\,,\; r_n\,,\; \ldots\,,$$

c as solution of $E_\infty(L) = A = 500\,.$

Density recursion III

If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$,

then the tails of the stopping time L behave like those of a geometric distribution,

i. e. $P_m(L > n + j) \approx \varrho^j \, P_m(L > n)$

with ϱ as eigenvalue with largest magnitude of the limit transition kernel.

$\rightsquigarrow$ use quadrature based methods until a certain n_0 and exploit the geometric tail,

e. g., $E_\infty(L) \approx \sum_{i=0}^{n_0-1} P(L > n) + \frac{P(L > n_0)}{1 - \tilde{\varrho}}$, $\quad \tilde{\varrho} = \frac{P(L>n_0)}{P(L>n_0-1)}$.

Remarks:

1. For homogeneous kernels Woodall (1983), Waldmann (1986) and Gan (1991) used the same ideas (exception D_m and D).
2. It is not clear, how in Margavio et al. (1995) the problem with the varying limits in the integration sequence is solved and why, e. g., Steiner (1999) did neither use nor even mention their results.

Density recursion IV

$$-a_n = b_n = c^* \qquad \texttt{fcl, stat, switch}\,,$$

$$-a_n = b_n = c^* \sqrt{1-(1-\lambda)^{2n}} = c^* \, l_n \qquad \texttt{vacl}\,,$$

$$a_n = -\big(1-(1-\lambda)^n\big)\, c^* \,,\; b_n = c^* \qquad \texttt{fir}\,,$$

$$a_n = -\big(1-(1-\lambda)^n\big)\, c^* \, l_n \,,\; b_n = c^* \, l_n \qquad \texttt{fvacl}\,,$$

$$-a_n = b_n = c^* \, l_n \left(1-(1-f)^{1+a(n-1)}\right) \qquad \texttt{fadj}\,.$$

$$\lim_{n\to\infty} -a_n = \lim_{n\to\infty} b_n = c^* \,.$$

$$r_n = (b_n - a_n)/(2c^*) \;,\; d_n = (a_n + b_n)/2 \,,$$

$$\begin{aligned} M_n(\tilde{z}, z) &= r_n M(d_{n-1} + r_{n-1}\tilde{z}, d_n + r_n z) \\ &= \frac{r_n}{\lambda}\phi_\mu\left(\frac{d_n + r_n z - (1-\lambda)(d_{n-1} + r_{n-1}\tilde{z})}{\lambda}\right) \;. \end{aligned}$$

$\rightsquigarrow$ smooth and positive (transition) kernel function on $[-c^*, c^*] \times [-c^*, c^*]$,

i. e. use abscissas independent on n and get simult. precise quadrature results

Critical values

in-control ARL, i. e. $E_\infty(L) = 500$, $\lambda = 0.1$

EWMA scheme	fcl	vacl	fir	fvacl	fadj	stat	switch
c	2.8143	2.8239	2.8415	2.8858	2.9131	2.8215	2.8879
$\widehat{E_\infty(L)}$, 10^9 rep.	499.99	500.02	500.01	499.92	500.02	500.01	499.94
se	.016	.016	.017	.017	.020	.016	.019
`density rec.`	499.99	500.04	499.99	499.93	500.04	499.99	499.97

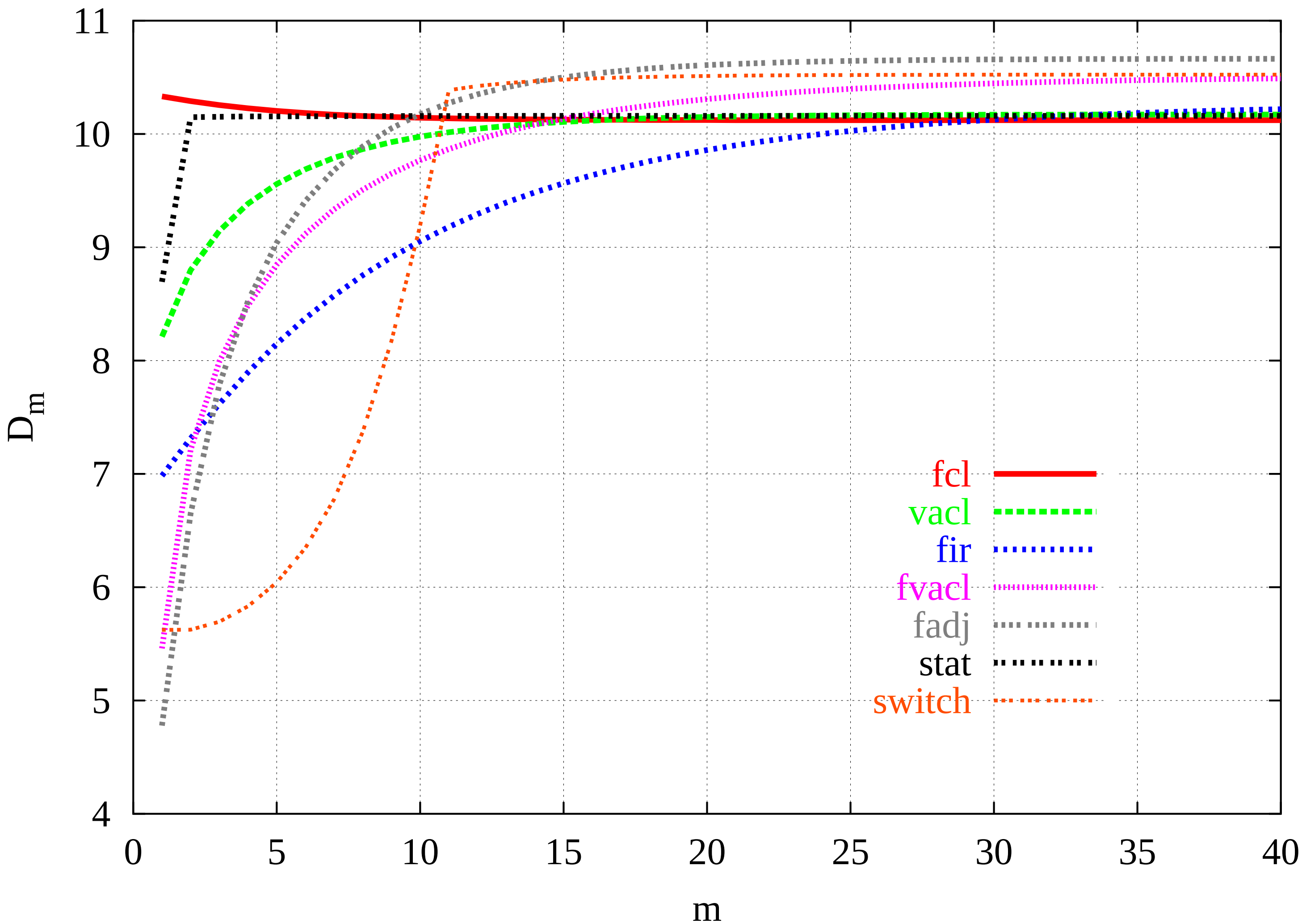
Zero-state and steady-state ARLs $E_1(L)$ and D

$\lambda = 0.1$, in-control ARL $E_\infty(L) = 500$, results as $\binom{E_1(L)}{D}$

δ	EWMA scheme fcl	vacl	fir	fvacl	fadj	stat	switch
0.5	**31.3**	28.8	24.8	22.9	21.6	29.3	**20.8**
	30.6	30.9	31.4	32.8	**33.6**	30.8	32.8
1.0	**10.3**	8.21	6.98	5.46	**4.78**	8.69	5.62
	10.1	10.2	10.3	10.5	**10.7**	10.2	10.5
1.5	**6.08**	4.17	3.90	2.52	**2.19**	4.56	3.36
	5.99	6.01	6.06	6.17	**6.24**	6.01	6.18
2.0	**4.36**	2.66	2.75	1.60	**1.45**	2.91	2.47
	4.31	4.32	4.35	4.42	**4.47**	4.32	4.43
3.0	**2.87**	1.51	1.81	1.09	**1.07**	1.57	1.68
	2.85	2.86	2.87	2.91	**2.94**	2.85	2.92

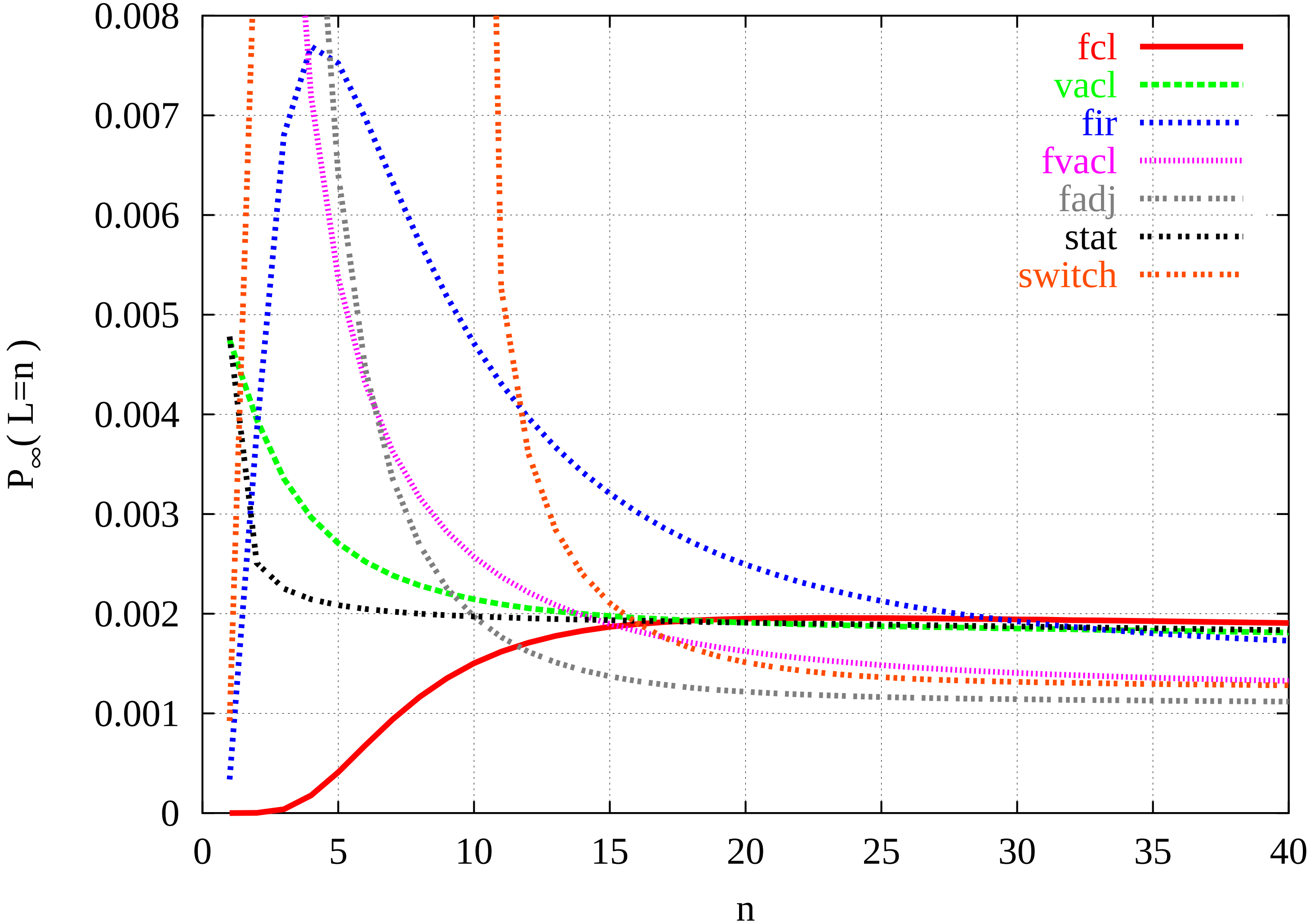
$$D_m = E_m(L - m + 1 | L \geq m)$$

$$(\lambda = 0.1,\ E_\infty(L) = 500,\ \delta = 1)$$



in-control PMF

$(\lambda = 0.1,\ E_\infty(L) = 500)$



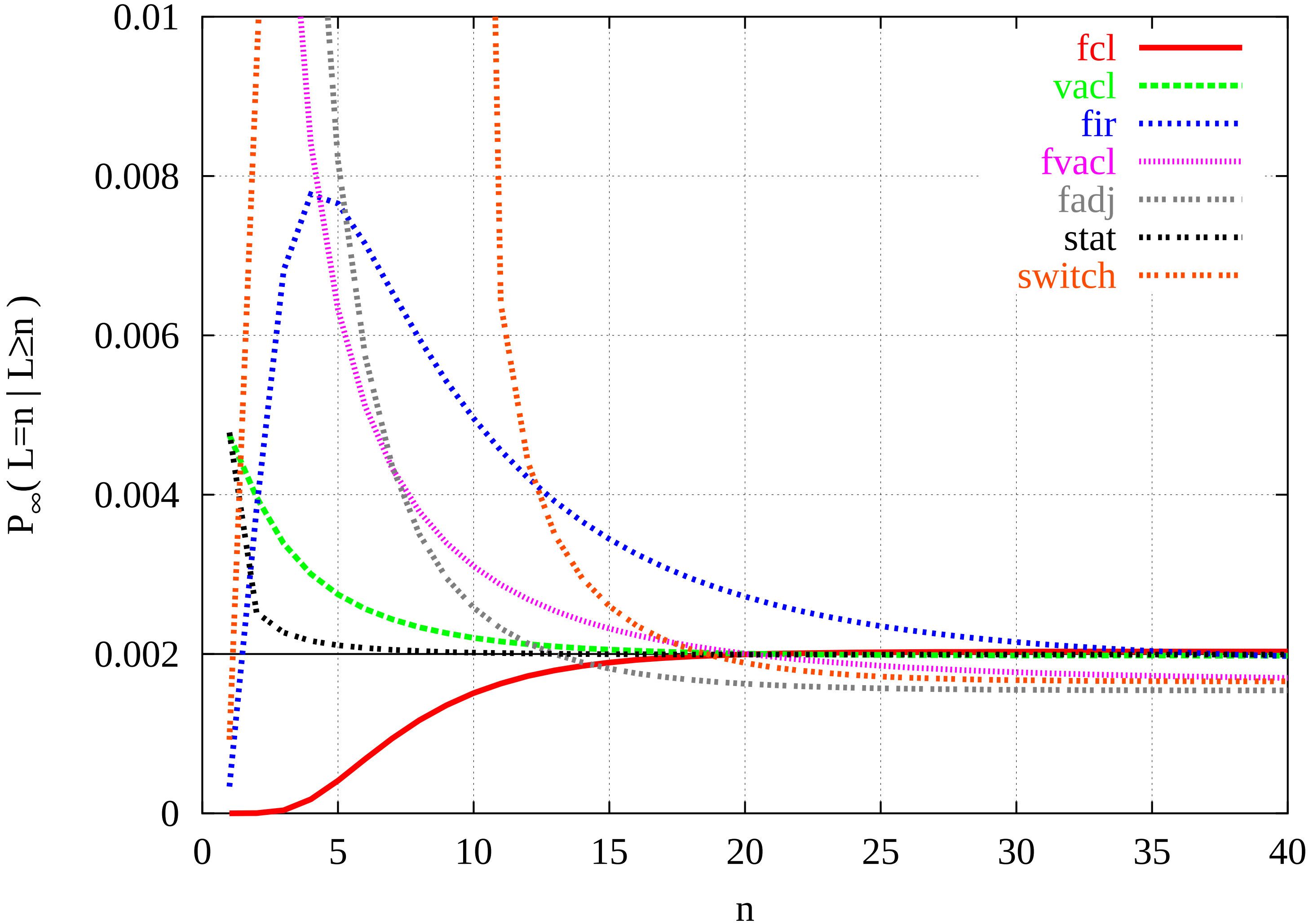
Probabilities for early false alarms

$\lambda = 0.1$, in-control ARL, i. e. $E_\infty(L) = 500$

EWMA scheme	fcl	vacl	fir	fvacl	fadj	stat	switch
$P_\infty(L = 1)$	0.0000	0.0047	0.0003	**0.1125**	**0.1452**	0.0048	0.0009
$P_\infty(L = 2)$	0.0000	0.0040	0.0038	0.0217	0.0435	0.0025	0.0094
$P_\infty(L = 3)$	0.0000	0.0034	0.0068	0.0109	0.0190	0.0022	0.0170
$P_\infty(L \leq 10)$	0.0063	0.0293	0.0551	**0.1742**	**0.2391**	0.0238	**0.1761**

false alarm rate $r_n = P_\infty(L = n | L \geq n)$

$(\lambda = 0.1,\ E_\infty(L) = 500)$



Conclusions

Regarding *fast initial response (FIR)*, one can distinguish three types of EWMA schemes:

feature	**"true" FIR**	**"moderate" FIR**	**balanced**
initial detection sensitivity	high	increased	low
long run performance	little deteriorated	slightly deteriorated	good
initial false alarm rate	high	moderate	low
	fvacl `(Steiner)`	fir `(Lucas/Sacc.)`	stat `(Knoth)`
	switch `(Fink)`		fcl
	fvacl `(Rhoads et al.)`		vacl