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Accurate ARL computation for variance monitoring schemes

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March 2004

Synopsis

1. Some definitions.
2. Drawbacks while computing ARLs for S^2 based schemes.
3. New approach.
4. A few side effects.
5. Resumé.

Some definitions

(i) Sequential change-point model,

(ii) Control charts,

(iii) ARL.

(i) Sequential change-point model

Modeling of a stochastic process with a possible distributional change

Sequence of random variables $X_1, X_2, \ldots$ with pdf $\{F_{(i)}\}$ and a certain (unknown) time point $m =$ **change-point** with

$$F_{(i)} = \begin{cases} F_0 & , \ i < m \\ F_1 & , \ i \geq m \end{cases} .$$

Example: $F_0 = \mathcal{N}(0,1)\,,\ F_1 = \mathcal{N}(1,1)$ + independence

Notation:

$\{X_i\}_{i=1}^{m-1}$ – process **in control**,

$\{X_i\}_{i=m}^{\infty}$ – process **out of control**.

(ii) Control charts

different names, same concepts:

change point detection, continuous inspection, surveillance, monitoring ...

Aim: Detect rapidly and reliably, whether there appeared change-point m!

- Transformation $\left\{X_i\right\}_{i=1,2,\ldots,n} \to Z_n$ and
- Stopping time $L = \min\left\{n \in \mathbb{N} : Z_n \notin \mathcal{O}\right\}$,

$$\mathcal{O} = (-\infty, ucl],\ [lcl, ucl],\ [lcl, \infty) \ldots$$

At time point L observation is stopped & the scheme signals an **alarm**.

Two control chart types under consideration

– the mean monitoring case.

- (1-sided) CUSUM: PAGE (1954)

$$Z_n = \max\left\{0, Z_{n-1} + X_n - k\right\},\ Z_0 = z_0 = 0\,,$$
$$L = \inf\left\{n \in \mathbb{N} : Z_n > h\right\} \qquad \big(k = (\mu_0 + \mu_1)/2\big)\,.$$

- (2-sided) EWMA: ROBERTS (1959)

$$Z_n = (1-\lambda) Z_{n-1} + \lambda X_n\,,\ Z_0 = z_0 = \mu_0\,,$$
$$L = \inf\left\{n \in \mathbb{N} : |Z_n - \mu_0| > c\sqrt{\lambda/(2-\lambda)}\right\}\,.$$

(iii) Average Run Length (ARL)

resembles the most popular performance measure.

Notation: $E_m(.)$ expectation for given change-point m.

Definition:

$$ARL = \begin{cases} E_\infty(L) & \text{, process in control} \\ E_1(L) & \text{, process out of control} \end{cases} .$$

Note that for dealing with the ARL, the sequence $\{X_i\}$ is (strong) stationary with the same probability law for all i. Thus, e. g.,

$$ARL = E_\mu(L) =: \mathcal{L}_\mu .$$

Established methods of ARL computation

(for control charts monitoring normal mean)

Look, e. g., at the CUSUM-ARL integral equation (PAGE, 1954):

$$\mathcal{L}(s) = 1 + F_X(k-s)\,\mathcal{L}(0) + \int_0^h f_X(z+k-s)\,\mathcal{L}(z)\,dz$$

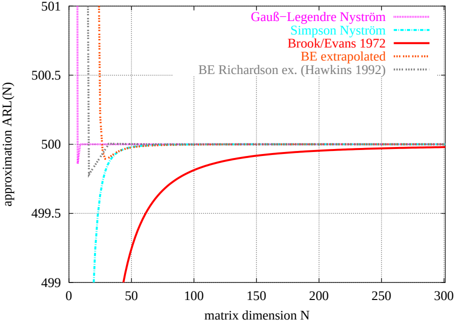
- $\mathcal{N}$, Markov chain approximation, BROOK/EVANS (1972), LUCAS/SACCUCCI (1990)

  (equivalent to Nyström method based on midpoint rule).
- *exponential*, exact, VARDEMAN/RAY (1985), GAN/CHANG (1998).
- $\mathcal{N}$, Gauß-Legendre Nyström method, VANCE (1986), CROWDER (1987).

Accuracy of different methods for ARL computation (mean)

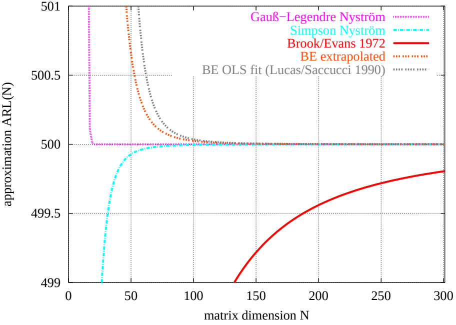
CUSUM with $k = 0.5\,,\ h = 4.389\,13$

(MC: 499.987 $\pm$ .016, 10^9 rep., 1450')



EWMA with $\lambda = 0.1\,,\ c = 2.814\,31$

(MC: 500.007 $\pm$ .016, 10^9 rep., 1400')



Variance monitoring and ARL computation

To begin with,

consider batches of size $N \geq 1$ and take

$$S_n^2 = \frac{1}{N-1} \sum_{j=1}^{N} \left(X_{nj} - \bar{X}_n\right)^2 \quad , \; \bar{X}_n = \frac{1}{N} \sum_{j=1}^{N} X_{nj}$$

$$\text{or} \quad \tilde{S}_n^2 = \frac{1}{N} \sum_{j=1}^{N} \left(X_{nj} - \mu_0\right)^2$$

as variance estimator V_n with df degrees of freedom.

Two control chart types under consideration – the variance monitoring case.

- (1-sided upper) CUSUM:

$$Z_n = \max\left\{0, Z_{n-1} + V_n - k\right\},\ Z_0 = z_0 = 0\,,$$
$$L = \inf\left\{n \in \mathbb{N} : Z_n > h\right\} \qquad \left(k = \frac{\sigma_1^2(\ln\sigma_1^2 - \ln\sigma_0^2)}{\sigma_1^2/\sigma_0^2 - 1}\right)\,.$$

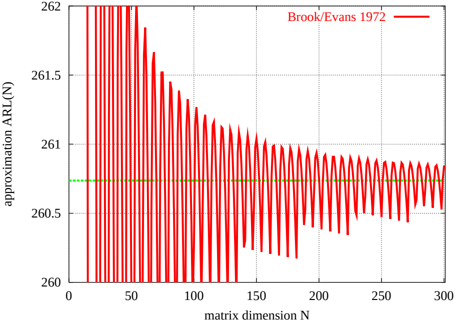
- (1-sided upper) EWMA:

$$Z_n = (1-\lambda)Z_{n-1} + \lambda V_n\,,\ Z_0 = z_0 = \sigma_0^2\,,$$
$$L = \inf\left\{n \in \mathbb{N} : Z_n > \sigma_0^2 + c\sqrt{\lambda/(2-\lambda)}\sqrt{2/df}\,\sigma_0^2\right\}\,.$$

Accuracy of different methods for ARL computation (variance)

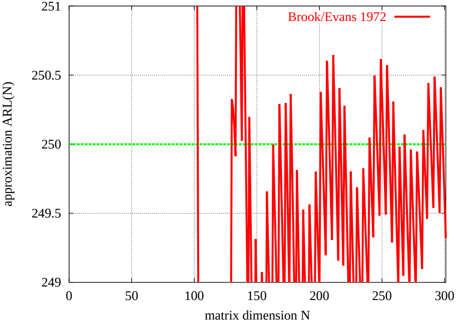
$$df = 1$$

CUSUM-S^2 with $k = 1.46\,,\ h = 10$



Ramírez/Juan 1989, BE, $N < 500$

EWMA-S^2 with $\lambda = 0.025\,,\ c = 1.661\,865$

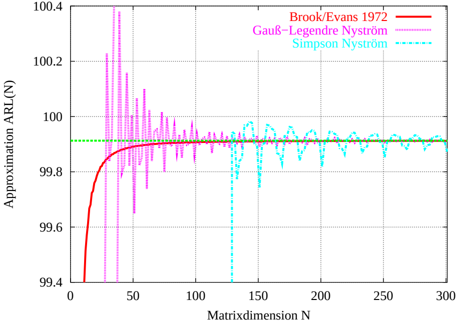


Domangue/Patch 1991, BE, $N \geq 200$

Accuracy of different methods for ARL computation (variance)

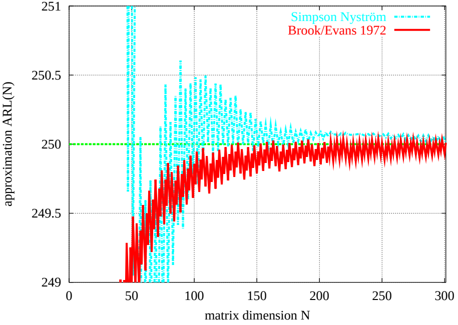
$$df = 4$$

CUSUM-S^2 with $k = 1.285\,,\ h = 2.921$



Chang/Gan 1995, MC, Acosta-Mejía et al. 1999, GL

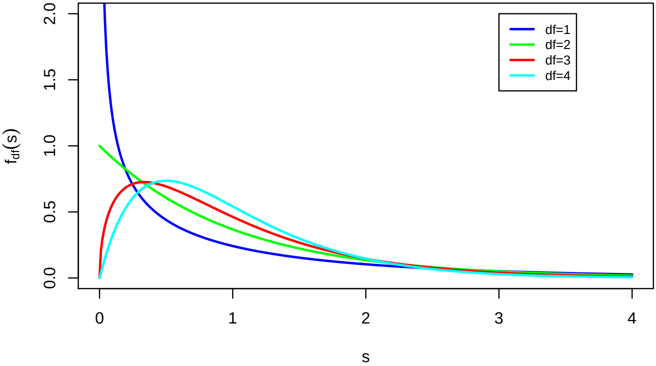
EWMA-S^2 with $\lambda = 0.18\,,\ c = 2.909\,223$



Mittag et al. 1998, BE, $N = 126$

Cause of the accuracy problem

is the behavior of the density of S^2 *at 0.*



New approach

What could we do?

- Product Nyström method: better than established ones, but worse than
- collocation.

Up to now, collocation for ARL computation was used by:

- Fellner (1990),
- Gianino/Champ/Rigdon (1990),
- Calzada/Scariano (2003).

(ARL of mean monitoring control charts only!)

Collocation for ARL computation of variance control charts

- Approximation of $\mathcal{L}(s)$ by linear combinations of
- (piecewise) Chebyshev polynomials (with additional exponential term for CUSUM),
- numerical quadrature of all integrals (less than 30 nodes are needed!),
- and solution of a linear equation system.

Sketch for the one-sided upper EWMA-S^2 control chart

$$c_u^* = \sigma_0^2 + c\sqrt{\lambda/(2-\lambda)}\sqrt{2/df}\,\sigma_0^2\,,$$

$$T_j(z) = \cos\left(j\arccos(z)\right)\,,\quad j = 0,1,\ldots,N-1\,,\ z\in[-1,1]\,,$$

$$T_j^*(z) = T_{j-1}\big((2z-c_u^*)/c_u^*\big)\,,\quad j = 1,2,\ldots,N\,,\ z\in[0,c_u^*]\,,$$

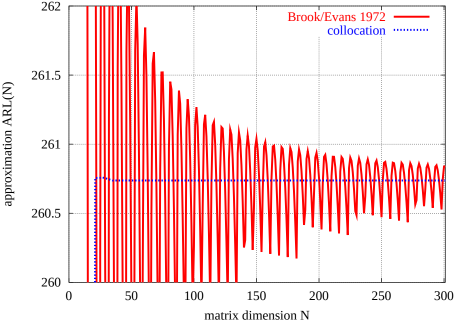
$$z_i = \frac{c_u^*}{2}\left[1+\cos\left((2i-1)/(2N)\,\pi\right)\right]\,,\quad i = 1,2,\ldots,N$$

$$\sum_{j=1}^{N} c_j\, T_j^*(z_i) = 1 + \sum_{j=1}^{N} c_j \int_{(1-\lambda)\,z_i}^{c_u^*} T_j^*(x)\,\frac{1}{\lambda}\,f_{df}\left(\frac{x-(1-\lambda)\,z_i}{\lambda}\right)\,dx\,.$$

Accuracy of different methods for ARL computation (variance)

$$df = 1$$

CUSUM-S^2 with $k = 1.46\,,\ h = 10$



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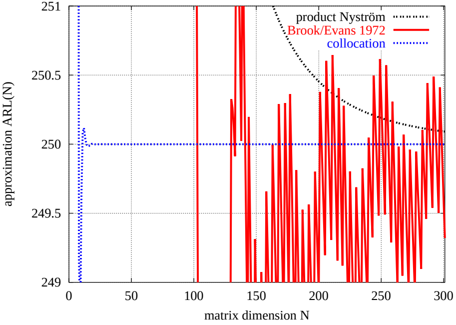

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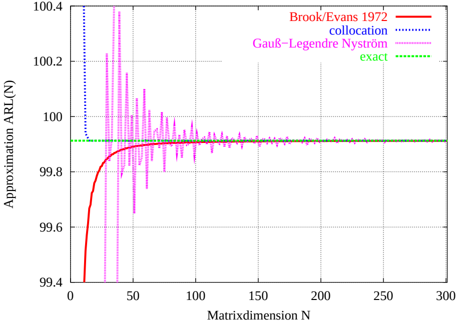

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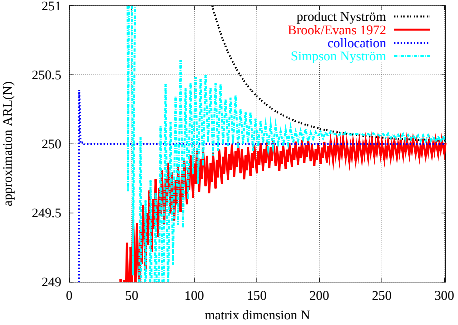

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Postscript for CUSUM-S^2 schemes

While χ^2_{df} random variables

with even degrees of freedom *df*

are Erlang distributed,

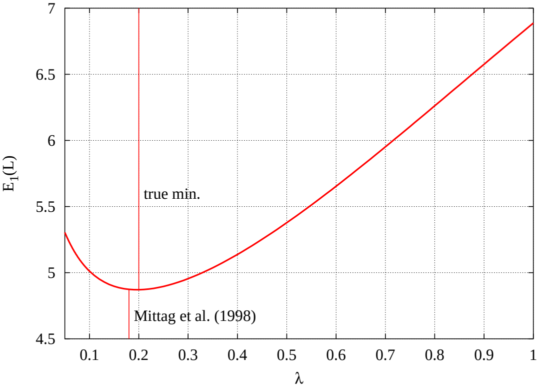
we could apply the results of Knoth (1998).

One-sided EWMA-S^2 chart for monitoring variance

$$Z_n = (1-\lambda) Z_{n-1} + \lambda S_n^2\,,\ n \ge 1\,,\ Z_0 = z_0 = \sigma_0^2 = 1\,,\ S_n^2 = 1/4 \sum_{j=1}^{5} (X_{nj} - \bar{X}_n)^2\,,$$

$$L = \inf\left\{ n \in \mathbb{N} : Z_n > \sigma_0^2 + c\sqrt{\lambda/(2-\lambda)}\sqrt{2/4}\,\sigma_0^2 \right\}\,,$$

$E_\infty(L) = 250\,,\ \sigma_1^2 = 1.5^2$ (as in Mittag et al., 1998).



$\lambda_{\min} = 0.197$

One-sided EWMA-S^2 chart for monitoring variance II

$$Z_n = (1-\lambda) Z_{n-1} + \lambda S_n^2 \,,\; n \geq 1 \,,\; Z_0 = z_0 = \sigma_0^2 = 1 \,,\; S_n^2 = 1/4 \sum_{j=1}^{5} (X_{nj} - \bar{X}_n)^2 \,,$$

$$L = \inf \left\{ n \in \mathbb{N} : Z_n > \sigma_0^2 + c\sqrt{\lambda/(2-\lambda)}\sqrt{2/4}\,\sigma_0^2 \right\} \,,$$

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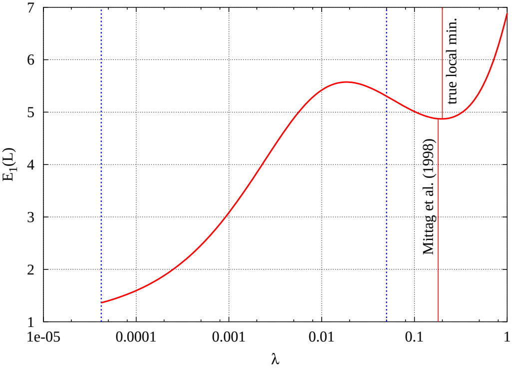

$\lambda = 0.000\,042$,

$c = 0.000\,064\,375\,308$,

$\widehat{E_\infty(L)} = 250.103 \pm 0.091$,

$\widehat{E_1(L)} = 1.3628 \pm 0.0000$,

10^9 rep.

$P_\infty(L = 1) \approx 0.4!$

One-sided EWMA-S^2 chart for monitoring variance III

$$Z_n = (1-\lambda) Z_{n-1} + \lambda S_n^2 \,,\; n \ge 1 \,,\; Z_0 = z_0 = \sigma_0^2 = 1 \,,\; S_n^2 = 1/4 \sum_{j=1}^{5} (X_{nj} - \bar{X}_n)^2 \,,$$

$$L = \inf \left\{ n \in \mathbb{N} : Z_n > \sigma_0^2 + c\sqrt{\lambda/(2-\lambda)}\sqrt{2/4}\,\sigma_0^2 \right\} \,,$$

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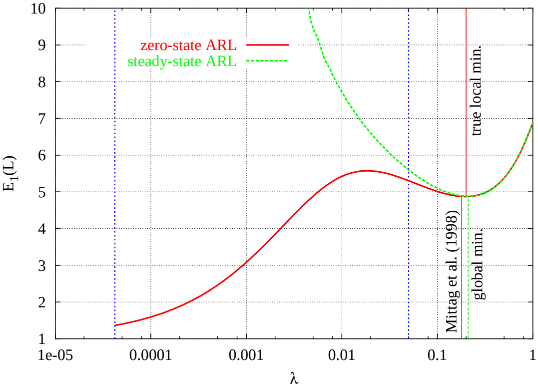

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EWMA: S^2 based vs. $\ln S^2$ based control charts

CROWDER/HAMILTON (1992)

Note:
Here, all schemes possess a reflecting barrier at σ_0^2 and $\ln \sigma_0^2$, resp.

	$\lambda = 0.05$		$\lambda = 0.16$		$\lambda = 0.32$	
σ^2	$\ln S^2$	S^2	$\ln S^2$	S^2	$\ln S^2$	S^2
1	200.00	200.00	200.00	200.00	200.00	200.00
1.1^2	**43.04**	37.78	45.59	43.44	48.93	50.50
1.2^2	**18.10**	16.71	18.54	17.42	19.63	20.05
1.3^2	10.75	10.32	**10.52**	9.85	10.73	10.74
1.4^2	7.63	7.39	7.20	6.68	**7.08**	6.93
1.5^2	5.97	5.74	5.49	5.03	5.24	5.03
2^2	3.17	2.74	2.77	2.33	2.44	2.18

Resumé

- Variance control charts were often applied, but the accuracy in computing their ARL values was poor.
- Collocation allows to cure these accuracy problems.
- Now, we are able to compute the ARL of EWMA-S^2 control charts for very small λ.
- It is not restricted to variance schemes. All distributions of chart statistics with restricted supports could be treated in this way.
- Collocation could be used for the steady-state ARL as well.
- Similar ideas could be used for (simultaneous) $\bar{X}$-S^2 schemes.

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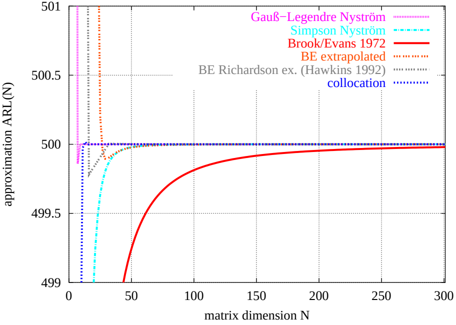
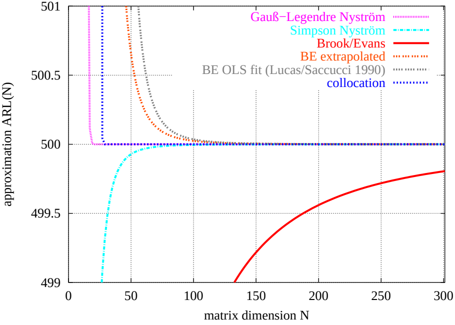
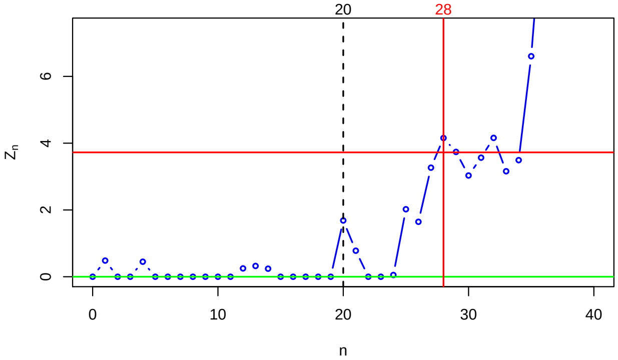




One-sided CUSUM-S^2 scheme



$$\sigma_0 = 1\,,\ \sigma_1 = 1.5\,,\ \text{batch size } N = 5\,,$$

$$Z_0 = 0\,,\ Z_n = \max\{0, Z_{n-1} + S_n^2 - k\}\,,$$

$$L = \min\{n \in \mathbb{N} : Z_n > h\}\,,$$

$$k = 1.46\,,\ h = 3.725\,,\ E_\infty(L) = 500\,,\ E_1(L) = 5.86\,.$$

Measuring control chart performance

- In order to evaluate "true" online procedures, first of all one has to take into account
  - elapsed time or
  - number of proceeded statistical objects.
- The control chart raises a signal with Prob. 1, whether there was a change or not.

$\rightsquigarrow$ Thus, assess properties of stopping time L!