Europa-Universität Viadrina Frankfurt (Oder)

CUSUMs and EMWAs – Favorites and Falsities

Sven Knoth
Dep. of Statistics, Europa–Universität Viadrina
knoth@euv-frankfurt-o.de

Stochastik-Tage Magdeburg

March 21, 2002

Outline

1. Introduction

2. Connections between worst case and average scenarios

3. Roberts' ARL study

4. Lucas'/Saccucci's worst case ARLs

(5. EWMA resembles the CUSUM for small λ)

1. Introduction

Change-point-model

sequence $X_1, X_2, \ldots$ with cdf $\{F_{\theta_{(i)}}\}_{i=1,2,\ldots}$ and unknown integer m = **change point** with

$$\theta_{(i)} = \begin{cases} \theta_0 & , \; i < m \\ \theta_1 & , \; i \geq m \end{cases} .$$

Control chart = detection, monitoring, or surveillance scheme

- transformation $\{X_i\}_{i=1,2,\ldots,n} \to Z_n$ and
- stopping time $L = \inf\left\{n \in \mathbb{N} : Z_n \notin \mathcal{O}\right\}$, $\mathcal{O} = (-\infty, ucl]$, $[lcl, ucl]$, or $[lcl, \infty)$.

Standard model

- independent $\{X_i\}$,
- $X_i \sim \mathcal{N}(\mu_{(i)}, \sigma^2 = 1)$,
- $\mu_{(i)} = \begin{cases} \mu_0 = 0 & , \; i < m \\ \mu_1 & , \; i \geq m \end{cases}$,
- $i < m$ – in control ..., $i \geq m$ – out of control state
- $E_m(\cdot)$ – expectation for change point at m
- Average Run Length – $ARL = E_\infty(L)$ and $ARL = E_1(L)$, respectively
  $\rightsquigarrow$ denote ARL with $\mathcal{L}_\mu$

One-sided Schemes

- CUSUM – Page (1954)

$$Z_n = \max\left\{0, Z_{n-1} + X_n - k\right\}, \; Z_0 = z_0 ,$$
$$L = \inf\{n \in \mathbb{N} : Z_n > h\} \qquad \big(k = (\mu_0 + \mu_1)/2\big)$$

- EWMA – Roberts (1959) (reflecting barrier – Waldmann (1986), Gan (1993))

$$Z_n = \max\left\{z^*_{\text{reflect}}, (1-\lambda)\, Z_{n-1} + \lambda\, X_n\right\}, \; Z_0 = z_0 ,$$
$$L = \inf\left\{n \in \mathbb{N} : Z_n > c\,\sqrt{\lambda/(2-\lambda)}\right\}, \; z^*_{\text{reflect}} = z_r\,\sqrt{\lambda/(2-\lambda)}$$

- GRSR – Girshick/Rubin (1952), Širjajev (1963/76), Roberts (1966)

$$Z_n = (1 + Z_{n-1})\exp(X_n - k)\,, \; Z_0 = z_0 ,$$
$$L = \inf\{n \in \mathbb{N} : Z_n > g\} \qquad \big(\exp[(\mu_1 - \mu_0)\, X_n - (\mu_1^2 - \mu_0^2)/2]\big)$$

Two-sided Schemes

- CUSUM: coupling of 1-sided schemes
- Crosier-CUSUM (1986)

$$Z_n = \begin{cases} 0 & , \; C_n \le k \\ (Z_{n-1} + X_n) \cdot \left(1 - \dfrac{k}{C_n}\right) & , \; C_n > k \end{cases}, \qquad C_n \underset{n \ge 1}{=} |\, Z_{n-1} + X_n \,| \,,$$

$$L = \inf\{n \in \mathbb{N} : |Z_n| > h\}$$

- EWMA – Roberts (1959)

$$Z_n = (1-\lambda)\, Z_{n-1} + \lambda\, X_n \,, \; Z_0 = z_0 \,, \; L = \inf\left\{n \in \mathbb{N} : |Z_n| > c\,\sqrt{\lambda/(2-\lambda)}\right\}$$

- GRSR: Pollak/Siegmund (1985) – $Z_n = \left(Z_n^+ + Z_n^-\right)/2$
  performs like coupling of 1-sided schemes $Z_n = \max\{Z_n^+, Z_n^-\}$ $\rightsquigarrow$ coupling
  (PS: $\mathcal{L}_1 = 11.142$, $E(D_*) = 9.644$, coupling: $\mathcal{L}_1 = 11.142$, $E(D_*) = 9.630$, both: $\mathcal{L}_0 = 500$)

Statements about performance

- Lorden (1971), Moustakides (1986), Ritov (1990): CUSUM is optimal
- Pollak (1987): GRSR is asymptotically optimal
- Lucas/Saccucci (1990): *Our comparisons showed that the ARL's for the EWMA are usually smaller than the ARL's of CUSUM up to a value of the shift near the one that the scheme was designed to detect.*
- different authors: little practical difference between ...

2. Worst case and average scenarios

Markov-type schemes

$$\mathcal{F}_{m-1} = \sigma\big(X_1, \ldots, X_{m-1}\big)\,,\ \mathcal{F}^*_{m-1} = \sigma\big(\mathcal{F}_{m-1} \cap \{L > m-1\}\big)\,,$$

$$D_*^{(m)} = E_m\big(L - m + 1 \,|\, \mathcal{F}^*_{m-1}\big) = \mathcal{L}_{\mu_1}(Z_{m-1})\,,\ Z_{m-1} \sim f^*_{m-1}(\cdot)$$

$$E_m\big([L - m + 1]^+ \,|\, \mathcal{F}_{m-1}\big) = D_*^{(m)} \cdot I_{\{L>m-1\}} + 0 \cdot I_{\{L \le m-1\}}\,,$$

$$\mathcal{D}_{\text{Lorden}} = \sup_{m\ge 1}\ \text{ess sup}\ \big(D_*^{(m)}\big)\,, \quad \mathcal{D}_{\text{Pollak/Siegmund}} = \sup_{m \ge 1}\ E\big(D_*^{(m)}\big)\,,$$

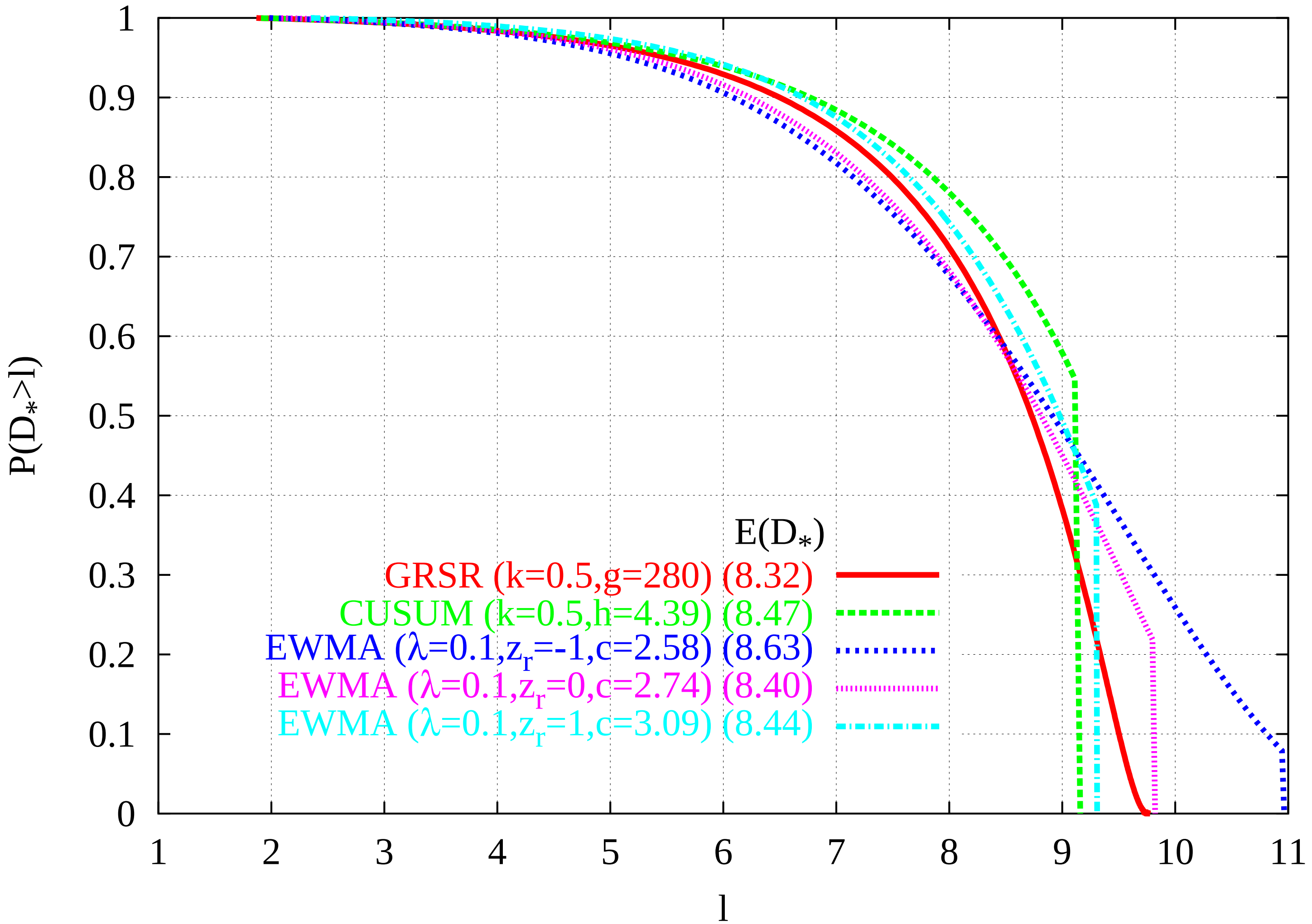
$$D_* = \lim_{m\to\infty} D_*^{(m)}\,,\ D_*^{(m)} \approx D_* \text{ for } m \ge m_* \text{ (small)}$$

$$E(D_*) = \int_{\mathcal{O}} \psi_{\mu_0}(z)\,\mathcal{L}_{\mu_1}(z)\,d\mathcal{M}(z)\,,$$

whereby $\psi_{\mu_0}(z) = \lim_{m\to\infty} f^*_{m-1}(z)$ with $f^*_{m-1}(z) := \dfrac{f_{m-1}(z)}{\int_{\mathcal{O}} f_{m-1}(z)\,d\mathcal{M}(z)}$

One-sided Schemes – survival function $P(D_* > l)$

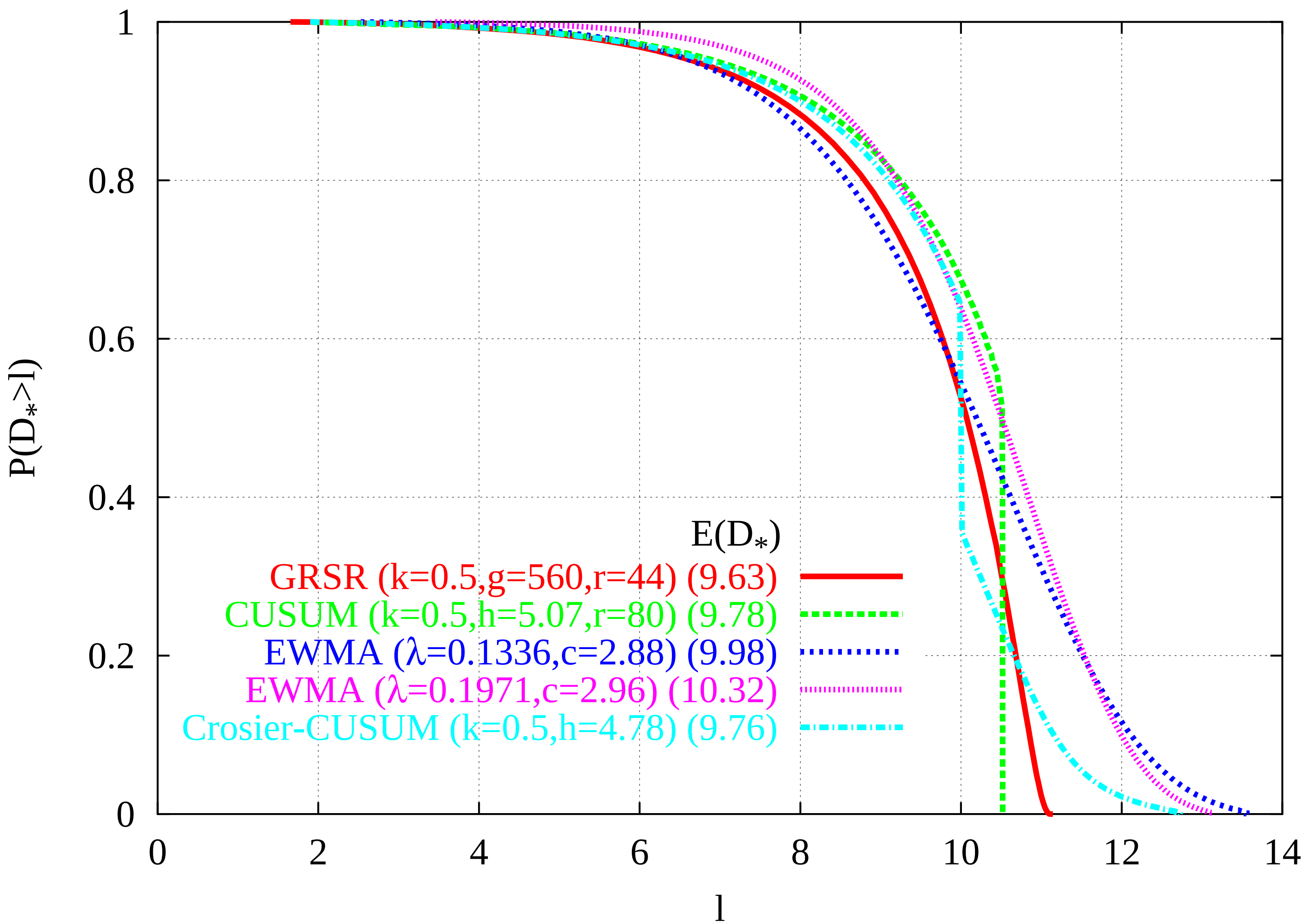
$E_\infty(L) = 500, \mu_1 = 1$



ess sup (D_*) : GRSR – 9.78, CUSUM – 9.16, EWMA$_{-1}$ – 10.96 (9.13), EWMA$_0$ – 9.82, EWMA$_1$ – 9.31

Two-sided Schemes – survival function $P(D_* > l)$

$E_\infty(L) = 500,\, \mu_1 = 1$



ess sup (D_*) : GRSR – 11.14, CUSUM – 10.52, EWMA$_1$ – 13.54 (10.19), EWMA$_2$ – 13.14 (10.52), Crosier – 12.73 (10.01)

3. Roberts did a study based on the ARL criterion

(one-sided schemes)

chart/scheme	**design ($\mathcal{L}_0 = 740$)**	**criterion**	**computation method**
Shewhart	$c = 3$	$\mathcal{L}_\mu = E(D_*)$	exact ($\mathcal{L}_0 = 741.43$)
MA	$n = 8$, $c = 2.79$	$E(D_*^{(9)})$	Monte Carlo, 25 000 repetitions
EWMA (GMA)	$\lambda = 0.25$, $c = 2.87$	$E(D_*^{(9)})$	Monte Carlo, 25 000 repetitions
CUSUM	$k = 0.47$, $h = 5$	$\mathcal{L}_\mu$	"translation" of two-sided results of Ewan/Kemp (1960)
GRSR	$k = 0.5$, $g = 390$	$E(D_*^{(9)})$	Monte Carlo, 50 000 repetitions

Remarks of Roberts

- **CUSUM:** *These curves all assume* $T = 0$*, and are upper bounds for those for* $T = 8$*, say, though the differences are small except for* $C_{26.8}(0)$*, ...*

  $\big(T = m - 1;\ C_{26.8}(0)$ — $h = 26.8, k = 0\big)$

- **MA/EWMA:** *By allowing the shifts to occur after eight observations with* $\mu_i = \mu_0$ *(i. e.,* $T = 8$*), complications due to changing limits were averted.*

  (EWMA: $\sqrt{Var(Z_9)/\big[\lambda/(2-\lambda)\big]} = \sqrt{1-(1-\lambda)^{2\cdot 9}} \underset{\lambda=0.25}{=} 0.997 \approx 1$)

Roberts – original results and recent values

scheme	Δ 0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	
$n = 8$	740	40	10.2	6.0	4.6	3.8	3.3	2.9	Roberts
MA	737	39	10.0	5.8	4.4	3.7	3.1	2.8	$E(D_*^{(9)})/E(D_*)$
$c = 2.79$	736	39	8.9	4.2	2.6	1.9	1.5	1.2	$E_1(L)$
$\lambda = 0.25$	740	40	10.1	5.1	3.5	2.7	2.2	1.9	Roberts
EWMA	686	39	9.8	5.0	3.4	2.6	2.1	1.8	$E(D_*^{(9)})/E(D_*)$
$c = 2.87$	681/689	37/39	8.4/10.0	3.9/5.1	2.5/3.4	1.9/2.6	1.5/2.2	1.3/1.9	$E_1(L)$
$k = 0.47$	740	34	10.0	5.8	4.3	3.5	2.9	2.5	Roberts
CUSUM	(719)/718	32	9.2/9.1	5.1	3.6	2.8	2.4/2.3	2.0	$E(D_*^{(9)})/E(D_*)$
$h = 5$	724	34	9.9	5.6	3.9	3.1	2.5	2.2	$E_1(L)$
$k = 0.5$	740	32	9.2	5.2	3.7	2.9	2.4	2.1	Roberts
GRSR	(690)/689	30	9.1/8.9	5.2/5.1	3.7/3.6	2.9/2.8	2.4	2.1	$E(D_*^{(9)})/E(D_*)$
$g = 390$	697	33	10.4	6.1	4.4	3.5	2.9	2.5	$E_1(L)$

4. Wrong worst case ARLs in Lucas/Saccucci 1990

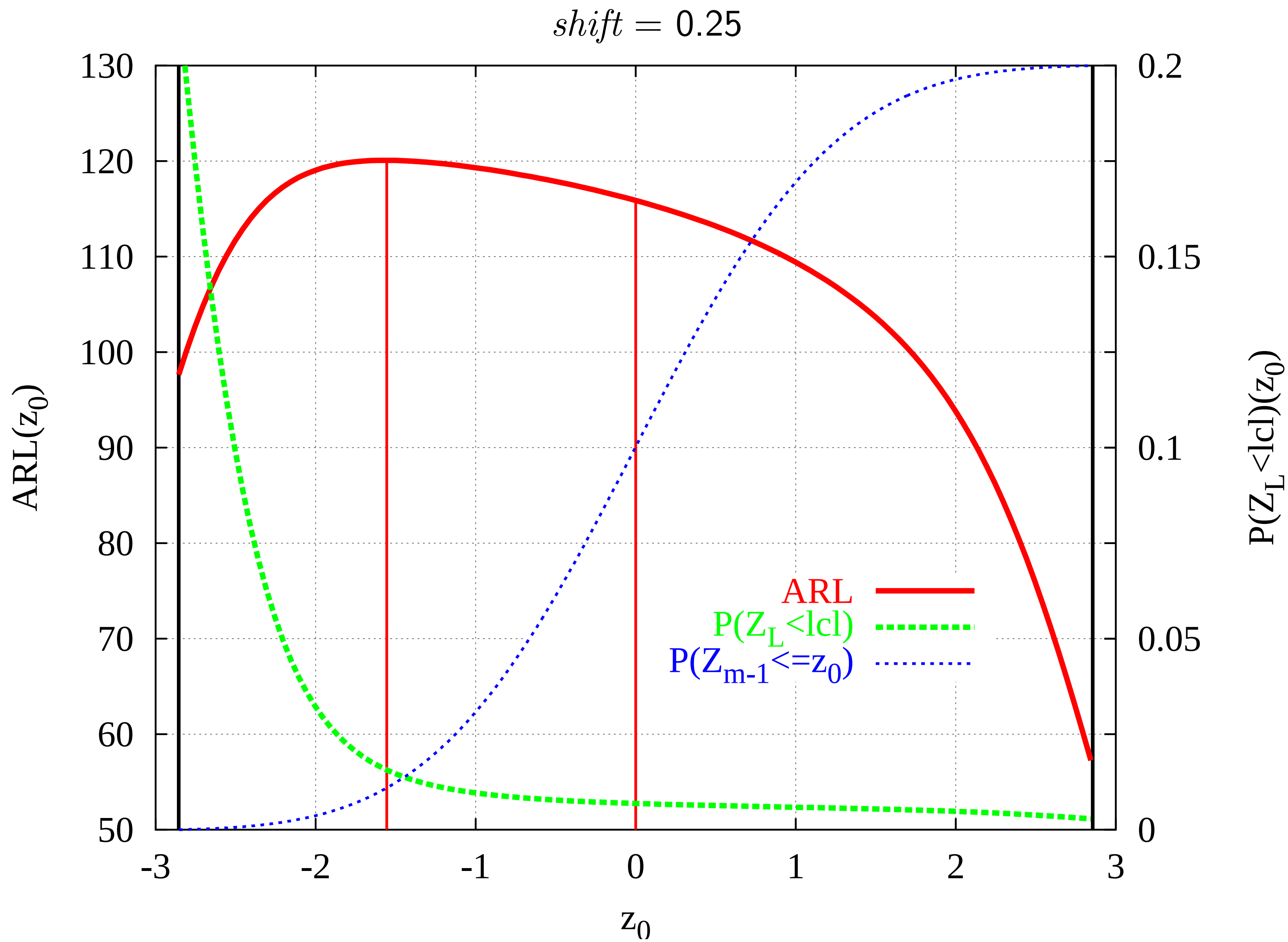
(two-sided schemes)

	Zero state			*worst case*		CUSUM*
shift	EWMA_1	EWMA_2	CUSUM	EWMA_1	EWMA_2	$k = 0.4,\ h = 6$
0.00	465	465	465	310 **(?)**	315 **(?)**	470
0.25	116_{114}	118_{117}	139_{137}	97.6 **(120)**	100 **(122)**	120_{116}
0.50	$33.3_{32.6}$	$33.8_{33.2}$	$38.0_{36.5}$	34.2 **(37.6)**	34.7 **(38.0)**	$33.6_{31.6}$
0.75	$16.0_{15.6}$	$16.1_{15.8}$	$17.0_{16.0}$	19.1 **(19.8)**	19.1 **(19.8)**	$16.5_{15.2}$
1.00	$10.1_{9.85}$	$10.0_{9.85}$	$10.4_{9.65}$	13.3 **(13.4)**	13.2 **(13.3)**	$10.6_{9.66}$
1.50	$5.71_{5.62}$	$5.67_{5.58}$	$5.75_{5.29}$	8.43	8.32	$6.19_{5.59}$

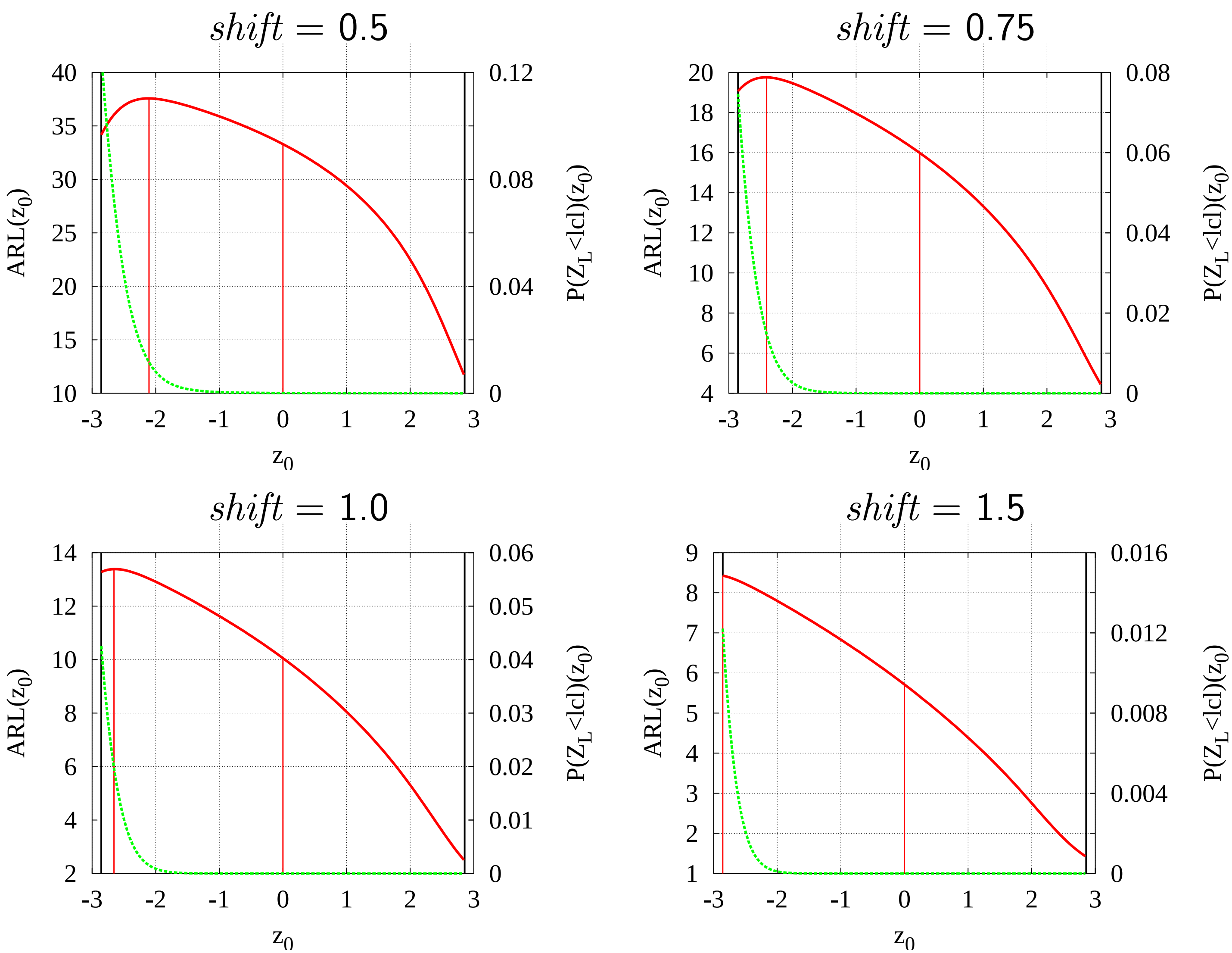
EWMA_1 – $\lambda = 0.133$, $c = 2.856$,
EWMA_2 – $\lambda = 0.139$, $c = 2.866$,
CUSUM – $k = 0.5$, $h = 5$

tiny values – $E(D_*)$

Worst case EWMA ARLs I

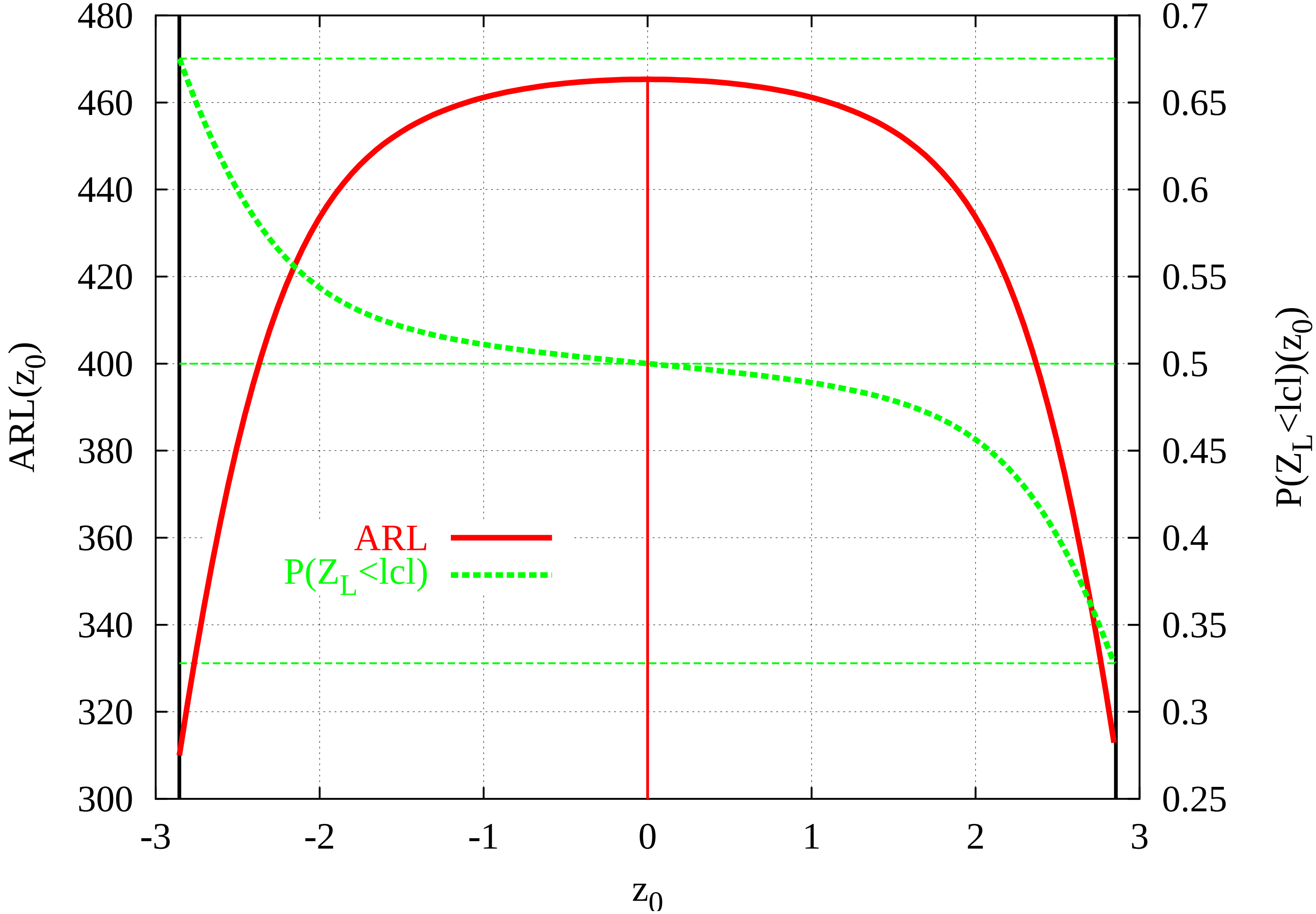


Worst case EWMA ARLs II



Worst case EWMA ARLs III

$shift = 0.0$ – in control



5. EWMA resembles the CUSUM for small λ

SAS Online-Manual (SAS 8.0, February 2000)

EWMAARL(δ,r,k)

r is the weight factor for the current subgroup mean in the EWMA, where $0 < r \leq 1$. If $r = 1$, the EWMAARL function returns the average run length for a Shewhart chart for means. Refer to Wadsworth and others (1986). If $r \leq 0.05$, $k \geq 3$, and $\delta < 0.10$, the algorithm used is unstable. However, note that the EWMA behaves like a cusum when $r \to 0$, and in this case the CUSUMARL function is applicable.

Hunter, J. S. (1986) *The exponentially weighted moving average*, JQT **18**, 203-210.

Thus, as $\lambda \to 0$, the EWMA takes on the appearance of the CUSUM. The EWMA control chart for values of $0 < \lambda < 1$ stands between the Shewhart and CUSUM control charts in its use of the historical data.

p. 207

EWMA resembles the CUSUM for small λ?

YES	**NO**
Crowder, S. V. (1987)	Woodall, W. H. & Maragah, H. D. (1990)
Mittag, H. J. (1993)	
Rinne, H. & Mittag, H. J. (1995)	**≈ NO**
Gohout, W. (1996)	Crowder, S. V. (1989)
Ledolter, J. (1999)	Hunter, J. S. (1990)
Stemann, D. (1997)	
SAS (2000)	**no comment**
Stemann, D. & Weihs, C. (2001)	many others

Hunter (1990)

HUNTER, J. S. (1990) *Discussion of Lucas & Saccucci*, Technometrics, **32**, p. 21-22.

Each observation in the CUSUM then takes weight $1/n_i$, where i is the index number of the trial. Of course, n_i varies randomly from trial to trial.

Consider $E(n_i) = ASN$ and $\sqrt{Var(n_i)} = SD(SN)$ and

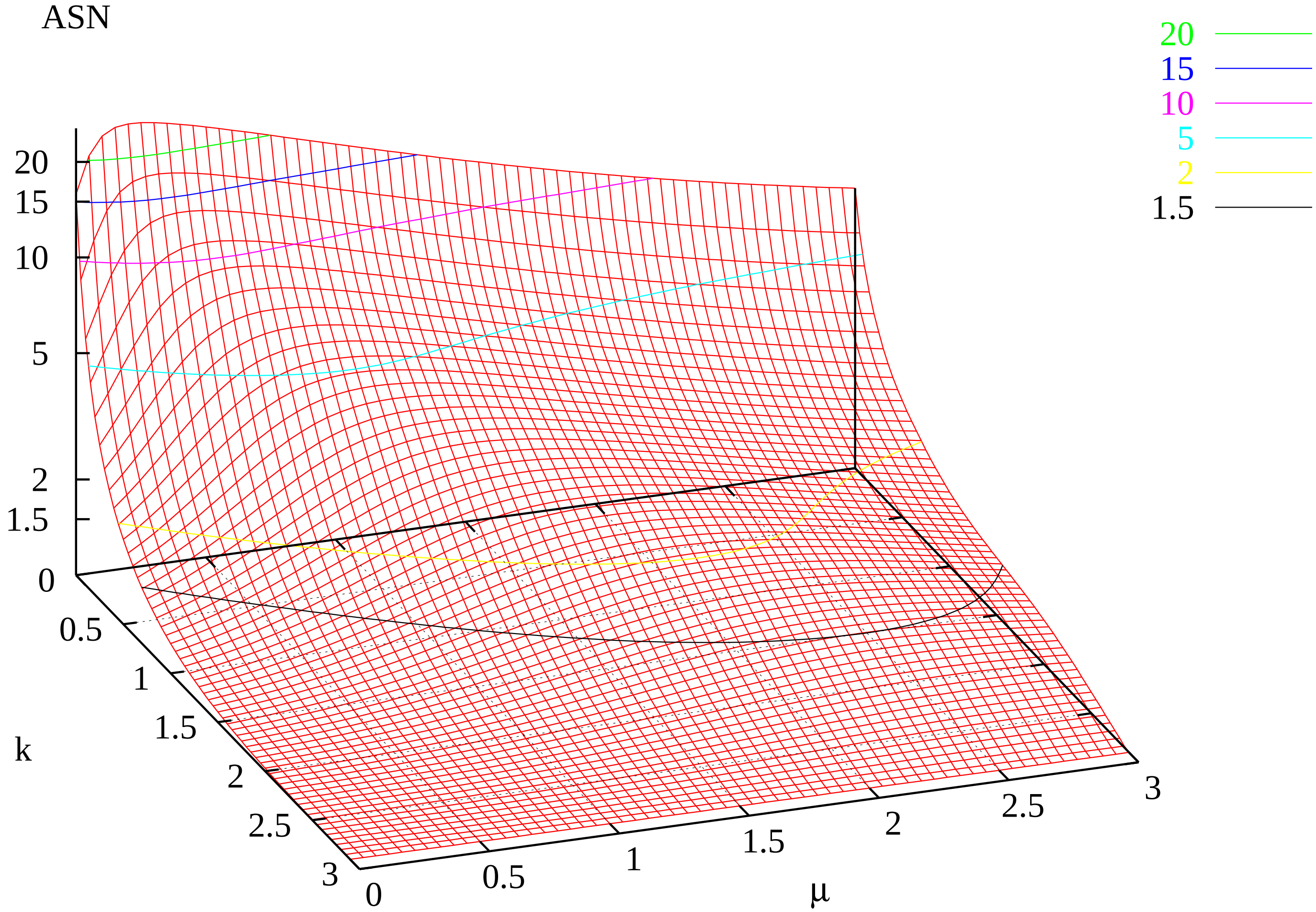
remember that $\mathcal{L}_\mu(z) = ASN_\mu(z) + OC_\mu(z)\,\frac{ASN_\mu(0)}{1-OC_\mu(0)}$

Example: one-sided CUSUM with $E_\infty(L) = 500$

$\rightsquigarrow$ "natural" domain of *reference value* k — $\Big[0, 2.8782 = 1/\big(1 - \Phi^{-1}(1/500)\big)\Big]$

Average Sample Number of (one-sided) CUSUM trials

$$E_\infty(L) = 500$$



SD of the Sample Number of (one-sided) CUSUM trials

$$E_\infty(L) = 500$$