More on control charting under drift

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X$^{\text{th}}$ International Workshop on ISQC
Seattle, August 2010

Outline

1. Introduction.
2. Drift models and schemes.
3. Numerical Algorithms.
4. Comparison of various schemes.
5. Conclusions.

Introduction

- Dominating model in SPC/change point detection/surveillance/... is the step change.
- In the ads language of practical SPC trend and step change are often confused.
- At least one type of runs rules was created especially for detecting trends: `Some number of successive points decrease or increase.`
- Statistical analysis of control charts under trend started in the 1980s: Bissell (1984), Asbagh (1985).
- Usually, linear trend (in observation number) is deployed.

Some history

198x – 199x

- BISSELL (1984): 1-sided, Shewhart, Shewhart with 2-of-2 runs rule, CUSUM, Markov chain.
- ASBAGH (1985): similar to BISSELL, more elaborated, better implemented, Shewhart-CUSUM added.
- SWEET (1988): 2-sided, coupled EWMA series (for mean and slope), none SPC performance measures calculated.
- DAVIS & WOODALL (1988): 2-sided, illustrate that (some) runs trend rule is useless, Monte-Carlo.
- GAN (1991,2): CUSUM (1-sided), EWMA (2-sided), accurate results.
- AERNE, CHAMP & RIGDON (1991): 2-sided, extensive comparison study, Markov chain, Monte-Carlo.

Some more history

1999 – 2009

- Chang & Fricker (1999): 1-sided, monotonically increasing means, detect asap level crossing and not the drift itself, CUSUM, EWMA, special GLR, Monte-Carlo.
- Reynolds & Stoumbos (2001): simultaneous EWMA (mean and variance — 2-/1-sided), Monte-Carlo.
- Fahmy & Elsayed (2006): 1-sided, local regression vs. Shewhart, CUSUM, EWMA, drift GLR, Monte-Carlo.
- Zou, Liu & Wang (2009): 1-sided, EWMA, CUSUM, GEWMA, step change & drift GLR, Monte-Carlo.

Change point models for drift

- $X_1, X_2, \ldots$ – sequence of independent random variables.
- Change point τ, drift coefficient Δ:

$$E(X_t) = \begin{cases} \mu_0 = 0 & , t < \tau \\ \mu_{t-\tau} = (t - \tau + 1)\Delta & , t \geq \tau \end{cases} .$$

(Gan preferred $\mu_{t-\tau} = (t - \tau)\Delta \quad \rightarrow \quad E(X_\tau) = 0 = \mu_0$)

$$Var(X_t) = \sigma_0^2 = 1 .$$

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- SWEET discussed treatment of non-equidistant sampling.
- CHANG & FRICKER considered $\mu_1 \leq \mu_2 \leq \ldots$ with some critical threshold δ.

Schemes/charts

1. Step change charts under drift
    - Shewhart,
    - Shewhart with runs rules (2 of 3, 4 of 5, ...)*,
    - CUSUM,
    - EWMA,
    - Shiryaev-Roberts (GRSR),
    - GEWMA,
    - step change GLR.

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   - GEWMA,
   - step change GLR.
2. Special drift charts
   - coupled EWMA (slope smoothing included),
   - local regression,
   - drift GLR,
   - step change charts to $diff(X)$.

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Schemes/charts

Some formulas for 1-sided setups – step change charts

Shewhart: $L = \inf\{n \in \mathbb{N} : X_n > c_S\}\,,$

CUSUM: $S_n = \max\{0, S_{n-1} + X_n - k\}\,,\; S_0 = s_0 = 0\,,$

$$L = \inf\{n \in \mathbb{N} : S_n > h\}\,,$$

EWMA: $Z_n = \max\left\{z^*_{\text{reflect}}, (1-\lambda)Z_{n-1} + \lambda X_n\right\},\; \lambda \in (0,1],\; Z_0 = \mu_0 = 0\,,$

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GEWMA†: $$\tilde{Z}_n(\lambda) = \sqrt{\frac{2-\lambda}{\lambda\left[1-(1-\lambda)^{2n}\right]}}\sum_{i=1}^{n}\lambda(1-\lambda)^{n-i}X_i\,,$$
$$L = \inf\{n \in \mathbb{N} : \max_{1\le k\le n} \tilde{Z}_n\big(1/k\big) > c_g\}\,.$$

† Han & Tsung (2004)

step change:

$$LR_n(\tau,\mu) = \prod_{i=\tau}^{n} \frac{e^{-[X_i-\mu]^2/2}}{e^{-X_i^2/2}} \to \max_{1\le\tau\le n,\,\mu}\,,$$

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drift:

$$LR_n(\tau,\Delta) = \prod_{i=\tau}^{n} \frac{e^{-[X_i-(i-\tau+1)\Delta]^2/2}}{e^{-X_i^2/2}} \to \max_{1\le\tau\le n,\,\Delta}\,,$$

$$\tilde{T}_n = \text{similar to step change}\,,$$

$$L = \inf\{n\in\mathbb{N} : \tilde{T}_n > h_D\}\,.$$

Schemes/charts

Local regression

- moving window of size w.
- OLS fit at the end of window n: $\hat{\mu}_{wn} = \hat{\alpha}_n + \hat{\beta}_n t_w$.

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- $M_n = \dfrac{(\mu_0 - \hat{\mu}_{wn})^2}{1/w + (t_w - \bar{t})^2/S_{tt}}$ with $S_{tt} = \sum_{i=1}^{w} (t_i - \bar{t})^2$.
- w/o drift $M_n \sim \chi^2_1$.

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- w/o drift $M_n \sim \chi^2_1$.
- neglect autocorrelation of M_n and treat it like a Shewhart chart:

$$L = \inf\{n \in \mathbb{N} : M_n > c_R\}\,.$$

- For details see Fahmy & Elsayed (2006).

Schemes/charts

Coupled EWMA schemes/Winter-Holt

- Two coupled EWMA series (S_t for mean, B_t for slope):

$$
\begin{aligned}
S_0 &= \mu_0 = 0\,,\\
S_t &= (1-\lambda_S)(S_{t-1}+B_{t-1}) + \lambda_S X_t\,,\\
B_0 &= \Delta_0 = 0\,,\\
B_t &= (1-\lambda_B)B_{t-1} + \lambda_B(S_t - S_{t-1})\,.
\end{aligned}
$$

- No usual performance study done so far.
- For details see Sweet (1988). He calculated adopted "control chart constants" to attain the usual chart behavior w/o checking it.

Control chart performance measures

Here (and there):

zero-state ARL $$\mathcal{L} = \begin{cases} E_1(L) & ,\tau = 1 \text{ ("early" change)} \\ E_\infty(L) & ,\tau = \infty \text{ (no change)} \end{cases} ,$$

steady-state ARL $$\mathcal{D} = \lim_{\tau\to\infty} E_\tau\big(L - \tau + 1 \mid L \geq \tau\big) .$$

Calculation

Notes

- Inhomogeneous transition kernel $\rightarrow$ nearly all classics fail.
- Most of the papers deploy Monte-Carlo studies.
- Bissell, Asbagh, Aerne et al. utilize Markov chain.
- Gan developed an iteration rule based on the classical (Nyström) solution of the ARL integral equation.
- Here also usage of Knoth (2003): yields $\mathcal{L}$ and $\mathcal{D}$.

Calculation

Markov chain

1. Approximate the continuous chart statistic with a finite Markov chain.
2. Denote $\mathbf{P}_n$ the transition matrix linked to transition $Z_{n-1} \to Z_n$, $\underline{z}_0$ the starting vector and $\underline{1}$ a vector of ones.
3. Then $\hat{P}(L > n) = \underline{z}_0' \left(\prod_{i=1}^{n} \mathbf{P}_i \right) \underline{1}$.
4. Thus, with N large (Cauchy Criterion like)

$$\hat{\mathcal{L}} = \sum_{n=0}^{N} \hat{P}(L > n)\,,$$

BISSELL and others

$$\hat{\mathcal{L}} = \sum_{n=1}^{N} n\big[\hat{P}(L > n-1) - \hat{P}(L > n)\big]\,.$$

For z within continuation region

$$
\begin{aligned}
f_1(z; z_0) &= \phi_{\mu_1}(z_0 \to z)\,, \\
f_n(z; z_0) &= \int f_{n-1}(\tilde{z}; z_0)\,\phi_{\mu_n}(\tilde{z} \to z)\,d\tilde{z}\,, \\
P(L > n)(z_0) &= \int f_n(\tilde{z}; z_0)\,d\tilde{z}\,,
\end{aligned}
$$

$+$ quadrature for integral.

More Details in KNOTH (2003).

Calculation

Gan (1991/2)

Denote $\mathcal{L}_j(y, \mu_j)$ the final ARL for a control chart starting at y and mean sequence $\mu_j = j\Delta, \mu_{j+1}, \ldots, \mu_m, \mu_m, \ldots$

$$\mathcal{L}_j(y, \mu_j) \underset{j=(0,)1,\ldots,m-1}{=} 1 + \int \mathcal{L}_{j+1}(y, \mu_{j+1}) \phi_{\mu_j}(y \to x)\, dx\,,$$

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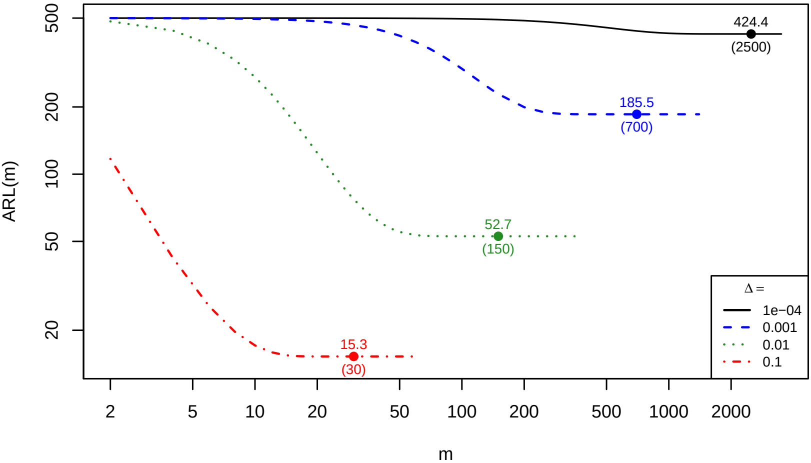
Choice of m: Perform the procedure for increasing m until $\mathcal{L}_1(y, \mu_1)$ remains constant. Gan (1991) reported

Δ	1	0.1	0.01	0.001	0.0001
m_{final}	6	30	150	700	2500

GAN (1991)

Choice of m — EWMA with $\lambda = 0.047$, $E_\infty(L) = 500$



Calculation

Gan (1991/2) vs. Knoth (2003)

- For sufficiently large m resp. N (and same quadrature) Gan and Knoth provide same accuracy.

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- Gradually increasing m in GAN's algorithm (his own approach) demands complete re-calculation of all $\mathcal{L}_j(y, \mu_j)$, while KNOTH's algorithm asks only for (some) new iterations. Thus, KNOTH's algorithm is faster.

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- Additionally, KNOTH's algorithm allows computation of $E_\tau(L - \tau + 1 \mid L \geq \tau)$ or arbitrary τ (thus also for very large ones).
- Eventually, $\mathcal{D}$ could be determined via $E_\tau(L - \tau + 1 \mid L \geq \tau)$ for $\tau \gg 1$ or the left eigenfunction of the transition kernel.

Bissell (1984/6)

1-sided CUSUM, $k = 0.5, h = 5$ in-control $\mathcal{L} \approx 930$:

Δ	$\hat{\mathcal{L}}^B_{MC}$	s. e.^{B}	B	B_{100}	B_{500}	GoK	$\hat{\mathcal{L}}_{MC}$	s. e.
0.001	245	23.8	–	95	229	231	231	0.113
0.002	142	12.4	–	92	155	156	156	0.068
0.005	87	7.0	205	81	89	89	89	0.033
0.01	56.4	3.8	101	57.1	57.1	57.2	57.2	0.019
0.02	36.2	2.1	53	36.5	36.5	36.5	36.5	0.010
0.05	18.8	1.2	24	20.4	20.4	20.4	20.4	0.005
0.1	14.3	0.58	14.2	13.3	13.3	13.3	13.3	0.003
0.2	8.4	0.32	9.0	8.8	8.8	8.8	8.8	0.002
0.5	5.2	0.12	5.3	5.3	5.3	5.3	5.3	0.001
1.0	3.40	0.13	3.6	3.60	3.60	3.60	3.60	0.001
2.0	2.44	0.10	2.5	2.50	2.50	2.50	2.50	0.000
3.0	1.96	0.04	2.0	2.01	2.01	2.01	2.01	0.000
# replicates	25						10^6	

Fahmy & Elsayed (2006)

2-sided CUSUM ($k = 0.25, h = 8$), EWMA ($\lambda = 0.1, c_E = 2.7$)

	CUSUM		EWMA		
Δ	FE (10^4)	(10^7)	FE (10^4)	(10^7)	GoK
0	368.333±0.549	368.251‡±0.111	365.749±3.598	369.021±0.114	368.994
0.10	13.986±0.026	14.086±0.001	12.971±0.029	12.986±0.001	12.986
0.25	8.560±0.014	8.656±0.000	7.738±0.015	7.758±0.000	7.758
0.50	5.946±0.008	6.033±0.000	5.312±0.009	5.318±0.000	5.318
0.75	4.827±0.007	4.898±0.000	4.279±0.007	4.286±0.000	4.285
1.00	4.156±0.006	4.224±0.000	3.680±0.006	3.688±0.000	3.688
2.00	2.950±0.003	2.989±0.000	2.598±0.005	2.616±0.000	2.616

‡368.394 would be the 'true' value.

Fahmy & Elsayed (2006)

FE's χ^2 (M_n), EWMA ($\lambda \in \{0.1, 0.2, 0.3, 0.5\}$), and trend GLR

	FE χ^2			EWMA				tGLR
Δ	$w^* = 3$	$w^* = 5$	$w^* = 20$	$\lambda = 0.1$	$\lambda = 0.2$	$\lambda = 0.3$	$\lambda = 0.5$	$w = 50$
0	379.138	370.048	373.458	369	370	370	370	367
0.10	17.445	16.047	12.860	12.986	**12.747**	13.041	14.136	13.229
0.25	8.537	8.127	7.623	7.758	7.304	**7.231**	7.497	7.380
0.50	5.027	4.869	5.260	5.318	4.881	4.722	**4.706**	4.758
0.75	3.672	3.673	4.250	4.285	3.886	3.715	**3.620**	3.665
1.00	**2.939**	3.055	3.660	3.668	3.318	3.149	3.023	3.055
2.00	**1.816**	2.042	2.579	2.616	2.254	2.124	2.005	1.964

2-sided CUSUM

Incidental remark

Already GAN mentioned that the idea of LUCAS/CROSIER (1982) to combine the ARL results of two one-sided CUSUM charts for getting the two-sided ARL does not work. All (GoK, Markov chain) algorithms do not converge for the chart opposite to the change.

Monte-Carlo studies must be utilized for ARL analysis of 2-sided CUSUM under trend.

Zou, Liu & Wang (2009)

1-sided schemes, EWMA, CUSUM, GRSR calculated with GoK, others by Monte-Carlo (10^4 rep.) in Zou et al.

	EWMA			CUSUM			GRSR			GEWMA	GLR-S	GLR-L
Δ	$\delta = 0.5$	$\delta = 1$	$\delta = 1.5$	$\delta = 0.5$	$\delta = 1$	$\delta = 1.5$	$\delta = 0.5$	$\delta = 1$	$\delta = 1.5$			
0	1750	1747	1733	1741	1742	1735	1730	1730	1730	-	-	-
0.0005	**318**	378	437	345	412	468	337	399	448	375	381	368
0.001	**215**	254	295	231	276	316	227	267	301	252	257	249
0.005	**83.5**	92.2	106	86.7	98.3	112	85.8	95.7	107	96.2	97.8	95.4
0.01	**55.7**	58.7	66.3	57.0	61.9	69.4	56.6	60.4	66.6	62.1	63.3	62.0
0.05	22.6	**21.1**	22.0	22.6	21.6	22.6	22.7	21.4	22.1	22.4	22.7	22.5
0.1	15.5	**13.9**	**13.9**	15.4	14.0	14.2	15.7	14.1	14.0	14.4	14.6	14.5
0.5	6.65	5.56	**5.09**	6.60	5.54	5.16	6.84	5.76	5.32	5.10	5.23	5.18
1.0	4.67	3.83	3.43	4.63	3.80	3.45	4.86	4.03	3.66	**3.26**	3.38	3.31
2.0	3.21	2.74	2.32	3.17	2.67	2.32	3.42	2.91	2.59	**2.09**	2.16	2.12
3.0	2.86	2.06	1.98	2.79	2.04	1.96	2.97	2.20	2.02	**1.69**	1.75	1.72
4.0	2.14	2.00	1.83	2.10	1.98	1.74	2.39	2.20	1.97	**1.31**	1.37	1.34
	$\lambda = 0.03$	$\lambda = 0.11$	$\lambda = 0.23$									

Zou, Liu & Wang (2009)

steady-state ARL

1-sided schemes, EWMA, CUSUM, GRSR calculated with Knoth

	EWMA			CUSUM			GRSR		
Δ	$\delta = 0.5$	$\delta = 1$	$\delta = 1.5$	$\delta = 0.5$	$\delta = 1$	$\delta = 1.5$	$\delta = 0.5$	$\delta = 1$	$\delta = 1.5$
0.0005	**314**	376	436	340	410	467	333	397	446
0.001	**213**	253	295	228	275	315	224	266	301
0.005	**82.6**	91.8	106	85.4	97.9	112	84.2	95.1	107
0.01	**55.1**	58.4	66.2	55.9	61.6	69.2	55.3	60.0	66.4
0.05	22.3	**20.9**	21.9	21.8	21.4	22.6	21.6	21.1	21.9
0.1	15.4	13.8	13.8	14.8	13.8	14.1	14.7	**13.7**	13.8
0.5	6.59	5.50	**5.05**	6.17	5.36	5.08	6.18	5.38	5.08
1.0	4.62	3.79	**3.40**	4.30	3.65	3.37	4.31	3.68	**3.40**
2.0	3.27	2.66	2.33	2.98	2.53	2.26	3.00	2.58	**2.30**
3.0	2.68	2.13	1.91	2.50	1.99	**1.90**	2.52	2.01	1.92
4.0	2.32	1.90	1.73	2.01	1.89	**1.66**	2.04	1.90	1.73

Exclusive drift monitoring

The naïve approach – *diff*(*X*)

If

$$E(X_t) = \begin{cases} \mu_0 = 0 & , t < \tau \\ \mu_{t-\tau} = (t - \tau + 1)\Delta & , t \geq \tau \end{cases} ,$$

then (equidistant time ...)

$$D_t \underset{t>1}{=} X_t - X_{t-1} \quad \text{and} \quad D_1 = X_1 - \mu_0 = X_1 \,,$$

$$E(D_t) = \begin{cases} 0 & , t < \tau \\ \Delta & , t \geq \tau \end{cases} \quad \text{with} \quad Var(D_t) = 2Var(X_t) = 2\sigma_0^2$$

and

$$Corr(D_t, D_{t-1}) = -1/2 \quad , \text{unfortunately.}$$

2-sided, for *diff*(*X*)-EWMA Monte-Carlo with 10^7 rep.

	diff(*X*)-EWMA				EWMA
Δ	$\lambda = 0.001$	$\lambda = 0.005$	$\lambda = 0.01$	$\lambda = 0.05$	$\lambda = 0.1$
0.000	$499.043_{0.158}$	$499.941_{0.158}$	$500.011_{0.158}$	$499.909_{0.158}$	500
0.001	$352.154_{0.082}$	$446.749_{0.134}$	$480.844_{0.150}$	$498.781_{0.157}$	200.366
0.005	$156.924_{0.024}$	$195.202_{0.037}$	$269.405_{0.068}$	$476.528_{0.150}$	81.377
0.010	$100.950_{0.013}$	$116.058_{0.017}$	$144.803_{0.027}$	$417.469_{0.130}$	53.341
0.050	$32.538_{0.003}$	$33.973_{0.005}$	$36.002_{0.004}$	$79.414_{0.018}$	20.028
0.100	$19.430_{0.002}$	$19.922_{0.002}$	$20.591_{0.002}$	$29.565_{0.004}$	13.343
0.500	$5.703_{0.000}$	$5.742_{0.000}$	$5.790_{0.000}$	$6.227_{0.001}$	5.439
1.000	$3.368_{0.000}$	$3.380_{0.000}$	$3.394_{0.000}$	$3.520_{0.000}$	3.768
2.000	$2.019_{0.000}$	$2.022_{0.000}$	$2.026_{0.000}$	$2.060_{0.000}$	2.688
3.000	$1.537_{0.000}$	$1.538_{0.000}$	$1.540_{0.000}$	$1.553_{0.000}$	2.047
4.000	$1.181_{0.000}$	$1.182_{0.000}$	$1.183_{0.000}$	$1.192_{0.000}$	1.993
5.000	$1.028_{0.000}$	$1.028_{0.000}$	$1.028_{0.000}$	$1.031_{0.000}$	1.927

Sweet 1988/Winter-Holt

Some first ARL results

Set $\lambda_B = \lambda_S = \lambda$:

$$S_0 = 0\,,\; S_t = (1-\lambda)(S_{t-1} + B_{t-1}) + \lambda X_t\,,$$
$$B_0 = 0\,,\; B_t = (1-\lambda)B_{t-1} + \lambda(S_t - S_{t-1})\,.$$

For $\lambda = 0.1$ and in-control $\mathcal{L} \approx 370$:

Δ	class. EWMA	Winter-Holt
0.001	177	211
0.026	28.6	30.4
0.081	14.7	15.4
0.325	6.72	7.08
	GoK	Monte-Carlo, 10^6

Conclusions

- Performance measures of step change control charts under drift could be calculated accurately (except the 2-sided versions of CUSUM and GRSR).
- It seems so that the additional (huge) complexity of the more sophisticated charts (GLR for step change and drift, GEWMA, local regression) are not worth the effort.
- The naïve approach does not work for very small drift coefficients such as $\Delta < 0.1$.
- Why do we consider such small drift coefficients (down to $\Delta = 0.0001$)? Nobody checks the detection performance for step change sizes smaller than $\delta = 0.1$.

Conclusions continued

- Clarification needed whether the drift itself is the target or the increasing mean (beyond a critical level)!
- Apply your favorite step change scheme and get a sufficiently powerful drift detection scheme.
- Study nonlinear trends and non-equidistant data series.
- Re-consider SWEET's coupled schemes.

Miscellaneous

ARLs of step change schemes under drift

In R library spc

In the recent version of the R library spc the following new functions are implemented:

- `xDewma.arl(l, c, delta, zr = 0, hs = 0, sided = "one",...`
- `xDcusum.arl(k, h, delta, hs = 0, sided = "one",...`
- `xDgrsr.arl(k, g, delta, zr = 0, hs = NULL, sided = "one",...`
- `xDshewhartrunsrules.arl(delta, c = 1, m = NULL, type = "12")`

Bissell (1984)

Recoded in R — preliminaries

```
transition.matrix <- function(k, h, mu, N) {
  i <- 1:(N-1)
  w <- 2*h/(2*N-1)
  qij <- function(i,j) pnorm((j-i)*w+w/2+k, mean=mu) -
                        pnorm((j-i)*w-w/2+k, mean=mu)
  Qi <- function(i) pnorm(-i*w+w/2+k, mean=mu)
  Q <- rbind( cbind( Qi(0), t(qij(0,i)) ),
              cbind( Qi(i), outer(i,i,qij) ) )
  Q
}

Bissel.ARL <- function(k, h, delta, N, m.max=1e3) {
  p.new <- ones <- rep(1, N)
  arl <- 1
  for ( m in 1:m.max ) {
    p.old <- p.new
    P <- transition.matrix(k, h, m*delta, N)
    p.new <- (P %*% p.old)[,1]
    arl <- arl + p.new[1]
  }
  arl
}
```

Bissell (1984)

Recoded in R — application

```
D.Bissel.ARL <- Vectorize(Bissel.ARL, "delta")

k <- .5
h <- 5
N <- 20

deltas <- c(.001, .002, .005, .01, .02, .05, .1, .2, .5, 1:3)

arls1e2 <- D.Bissel.ARL(k, h, deltas, N, m.max=1e2)
arls5e2 <- D.Bissel.ARL(k, h, deltas, N, m.max=5e2)
arls1e3 <- D.Bissel.ARL(k, h, deltas, N, m.max=1e3)
arls5e3 <- D.Bissel.ARL(k, h, deltas, N, m.max=5e3)

print(round(data.frame(deltas, arls1e2, arls5e2, arls1e3, arls5e3), digits=3))
```

Bissell (1984)

Recoded in R — results

	deltas	arls1e2	arls5e2	arls1e3	arls5e3
1	0.001	94.649	229.485	229.638	229.638
2	0.002	92.413	155.464	155.464	155.464
3	0.005	80.870	88.865	88.865	88.865
4	0.010	57.078	57.102	57.102	57.102
5	0.020	36.508	36.508	36.508	36.508
6	0.050	20.382	20.382	20.382	20.382
7	0.100	13.316	13.316	13.316	13.316
8	0.200	8.838	8.838	8.838	8.838
9	0.500	5.258	5.258	5.258	5.258
10	1.000	3.605	3.605	3.605	3.605
11	2.000	2.498	2.498	2.498	2.498
12	3.000	2.010	2.010	2.010	2.010