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Simultaneous EWMA charts for controlling mean and variance in the presence of autocorrelation

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1. Simultaneous charts and autocorrelation

2. Modified EWMA charts

3. EWMA residual charts

4. Comparison and conclusions

simultaneous charts for mean and variance

Montgomery (1991),

Gan (1995),

Kanagawa and Arizono (1997),

Mittag and Stemann (1997)

control charts and autocorrelated data

Goldsmith and Woodward (1961),

Bagshaw and Johnson (1975),

Nikiforov (1975/79),

Vasilopoulos and Stamboulis (1978),

Alwan (1989),

MacGregor and Harris (1993),

Amin, Schmid, and Frank (1997),

198x, 199x ...

simultaneous & correlation

(Lu and Reynolds, 1999)

Knoth, Schmid, and Schöne (1998): $\bar{X}$–S^2 and $\bar{X}$–R chart

$\Downarrow$

EWMA ?

Change point model

target process $\{Y_{i,j}\}$, observed $\{X_{i,j}\}$

i – sample number, j number within the sample (sample size n)

$$X_{i,j} = \begin{cases} Y_{i,j} & , i < q \\ \mu_0 + \Delta\big(Y_{i,j} - \mu_0\big) + a\sqrt{\gamma_0} & , i \geq q \end{cases}$$

with change point q and
$\mu_0 = E(Y_{i,j})$, $\gamma_0 = Var(Y_{i,j})$.

$\rightsquigarrow$

$$E(X_{i,j}) = \begin{cases} \mu_0 & , i < q \\ \mu_0 + a\sqrt{\gamma_0} & , i \geq q \end{cases}$$

$$Var(X_{i,j}) = \begin{cases} \gamma_0 & , i < q \\ \Delta^2 \gamma_0 & , i \geq q \end{cases}$$

EWMA chart (iid)

$$Z_{\bar{X},i} \underset{i\geq 1}{=} (1-\lambda_1)\, Z_{\bar{X},i-1} + \lambda_1\, \bar{X}_i\,,$$

$$Z_{\bar{X},0} = E_0(\bar{X}) = \mu_0\,,$$

$$Z_{S^2,i} \underset{i\geq 1}{=} (1-\lambda_2)\, Z_{S^2,i-1} + \lambda_2\, S_i^2\,,$$

$$Z_{S^2,0} = E_0(S^2) = \gamma_0\,,$$

$$\tau = \inf \left\{ i \in I\!N : \left| Z_{\bar{X},i} - \mu_0 \right| > x_{iid} \sqrt{\frac{\lambda_1}{2-\lambda_1}}\, \sqrt{Var_0(\bar{X})} \right.$$

$$\left. \text{or} \qquad Z_{S^2,i} - \gamma_0 \; > s_{iid} \sqrt{\frac{\lambda_2}{2-\lambda_2}}\, \sqrt{Var_0(S^2)} \right\}$$

(one–sided for the variance !!)

Average Run Length – ARL

- most popular performance measure of control charts

"expectation of the stopping time τ"

1. in–control
   $\leftrightarrow q = \infty$ or $a = 0$ and $\Delta = 1$:

   $E_0(\tau)$

2. out–of–control
   $\leftrightarrow q = 1$ and $a \neq 0$ or $\Delta \neq 1$:

   $E_1(\tau)$

further:
conditional, steady–state delays, quantiles ...

Autocorrelation model

($Y_{i,j}$ – target process)

independence between samples

$$\mathbf{Y}_i = (Y_{i,1}, Y_{i,2}, \ldots, Y_{i,n})' \quad , \; i = 1, 2, \ldots$$

dependence within the sample: $AR(1)$

$$Y_{i,j} \;=\; \mu_0 + \alpha \left(Y_{i,j-1} - \mu_0 \right) + \varepsilon_{i,j} \, ,$$

$$j = 1, 2, \ldots, n \, ,$$

$$|\alpha| < 1 \, ,$$

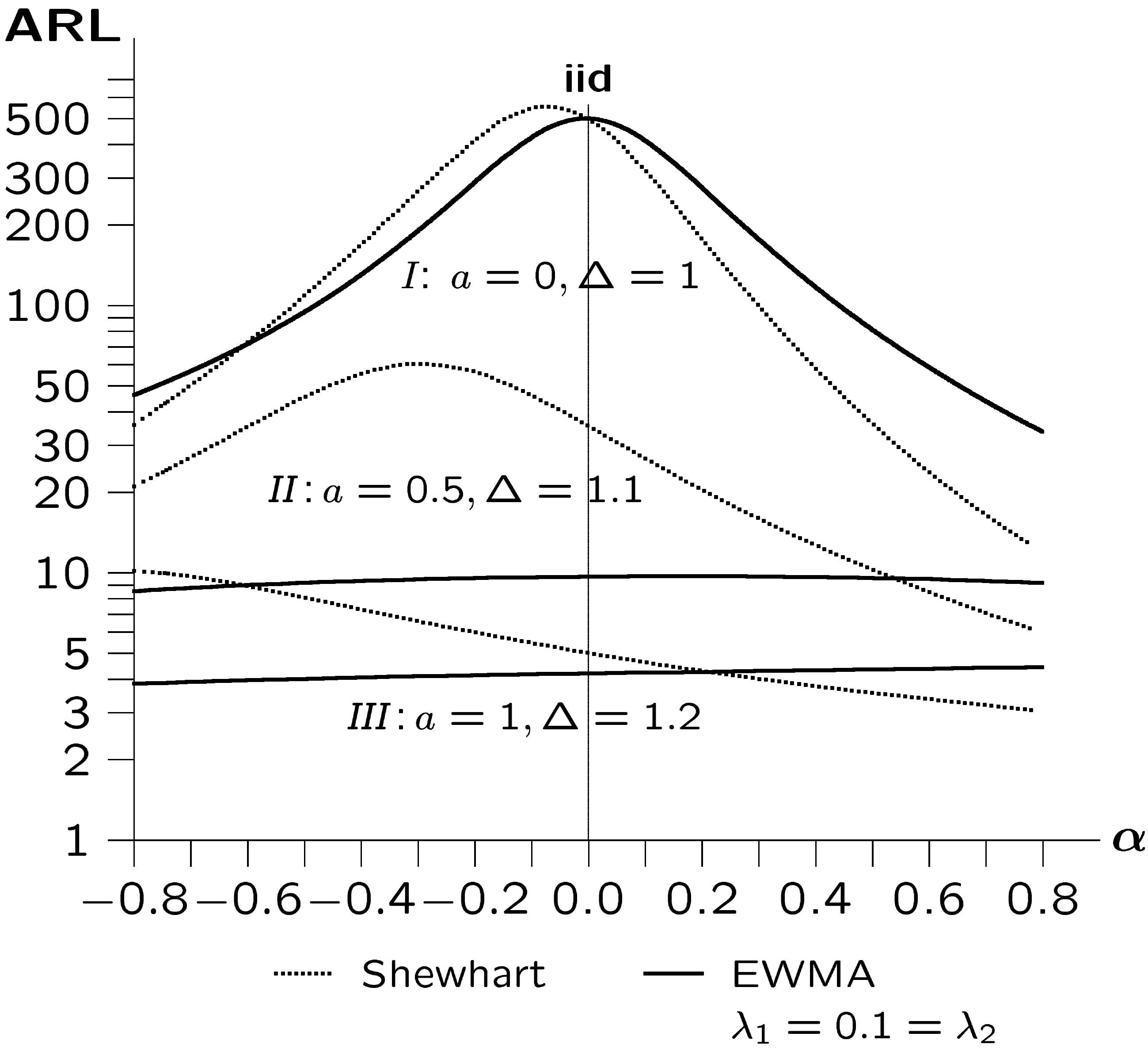
$$\varepsilon_{i,j} \sim \mathcal{N}(0, \sigma_0^2) \quad \text{(iid)} \, .$$

$\leadsto$

$$E(Y_{i,j}) \;=\; \mu_0 \, ,$$

$$Var(Y_{i,j}) \;=\; \frac{\sigma_0^2}{1 - \alpha^2} \;=\; \gamma_0 \, .$$

Influence of the autocorrelation coefficient α on the ARL



Main concepts of control charts for correlated data

start with fit of appropriate time series model – here $AR(1)$

modified charts

adapt
variance and critical value

$\rightarrow$ correct in–control ARL

$\Downarrow$

transformed threshold for original data

residual charts

model residuals are iid and

the empirical residuals as well (starting problems)

$\Downarrow$

classical charts on transformed data

special case:

cusum charts of Nikiforov (1975/79), Schmid (1997)

Modified $\bar{X}$–S^2 EWMA chart I

adapted moments

with autocorrelation function

$$\gamma_h = Cov_0(X_{i,j}, X_{i,j+h})$$

$$Var_0(\bar{X}) = \frac{1}{n} \sum_{|j|<n} \left(1 - \frac{|j|}{n}\right) \gamma_j \,,$$

$$E_0(S^2) = \gamma_0 - \frac{2}{n-1} \sum_{j=1}^{n-1} \left(1 - \frac{j}{n}\right) \gamma_j \,,$$

$$Var_0(S^2) =$$

$$\frac{2}{(n-1)^2} \left[\sum_{v,j=1}^{n} \gamma_{v-j}^2 - \frac{2}{n} \sum_{v=1}^{n} \left(\sum_{j=1}^{n} \gamma_{v-j} \right)^2 + \frac{1}{n^2} \left(\sum_{v,j=1}^{n} \gamma_{v-j} \right)^2 \right]$$

(assuming normality)

$$AR(1): \qquad \gamma_h = \frac{\alpha^{|h|}\,\sigma_0^2}{1-\alpha^2}$$

Modified $\bar{X}$–S^2 EWMA chart II

adapted critical values

prespecify in–control ARL $\to$ determine related critical values (x_m, s_m)

1. approximate ARL using Markov chain approach (Brook/Evans 1972):
   - 2dimensional chain
   - transition probabilities:
     joint distribution of $(\bar{X}, S^2)$
     $\leadsto$ Schöne/Schmid (2000)

$$P_{\bar{X},S^2}\big(|\bar{X}| \le x, S^2 \le s\big)$$

$$= \sum_{i,j=0}^{\infty} \frac{f_{i+1,j}}{\sqrt{2\pi}} \frac{4\beta}{2j+1} g_i^{(n+1)}(s)\, x^{2j+1}$$

$$+ \chi^2_{n-1}\left(\frac{s}{\beta}\right) \left[\Phi\left(\frac{x}{\sqrt{b_0}}\right) - \Phi\left(\frac{-x}{\sqrt{b_0}}\right)\right]$$

2. additional condition:

   univariate charts – same ARLs

3. two nonlinear equations,
   two unknown parameters

   $\rightarrow$ solution by 2dimensional secant rule

4. In study:

   maximal dimension $\approx 1600 = 40 \cdot 40$.

$\bar{X}$–S^2 EWMA residual chart I

standardized residuals – AR(1)

(X – observed, Y – target)

$$
\begin{aligned}
\widehat{\varepsilon}_{i,j} &= \frac{X_{i,j} - \widehat{X}_{i,j}}{\sqrt{Var_0\big(X_{i,j} - \widehat{X}_{i,j}\big)}}\,, \\
\widehat{X}_{i,j} &= \begin{cases} \mu_0 & , j = 1 \\ \mu_0 + \alpha\,\big(X_{i,j-1} - \mu_0\big) & , j > 1 \end{cases}\,, \\
Var_0\big(X_{i,j} - \widehat{X}_{i,j}\big) &= \begin{cases} \gamma_0 & , j = 1 \\ \sigma_0^2 & , j > 1 \end{cases}\,, \\
X_{i,j} - \widehat{X}_{i,j} &= \begin{cases} \Delta\,\big(Y_{i,1} - \mu_0\big) + a\,\sqrt{\gamma_0} & , j = 1 \\ \Delta\,\varepsilon_{i,j} + a\,\sqrt{\gamma_0}\,(1-\alpha) & , j > 1 \end{cases}\,, \\
\bar{\bar{\varepsilon}}_{i,j} &= \frac{1}{n}\sum_{j=1}^{n}\widehat{\varepsilon}_{i,j}\ ,\ \widehat{S}_i^2 \ = \ \frac{1}{n-1}\sum_{j=1}^{n}\big(\widehat{\varepsilon}_{i,j} - \bar{\bar{\varepsilon}}_{i,j}\big)^2\,, \\
E_0(\bar{\bar{\varepsilon}}_{i,j}) &= 0 \quad , \ Var_0(\bar{\bar{\varepsilon}}_{i,j}) \ = \ 1/n\,, \\
E_0(\widehat{S}_i^2) &= 1 \quad , \ Var_0(\widehat{S}_i^2) \ = \ 2/(n-1)\,.
\end{aligned}
$$

$\bar{X}$–S^2 EWMA residual chart *II*

computation of the ARL

1. Gan (1995): $\sim$ Waldmann (1986)
   Runger and Prabhu (1996)

2. here:

$$
\begin{aligned}
E_0(\tau_{biv}) &= \sum_{i=0}^{\infty} P_0(\tau_{biv} > i) \\
&= \sum_{i=0}^{\infty} P_0(\tau_1 > i) \times P_0(\tau_2 > i) \\
&\approx \sum_{i=0}^{\infty} \mathbf{p}_1' \, \mathbf{P}_1^i \, \mathbf{1} \times \mathbf{p}_2' \, \mathbf{P}_2^i \, \mathbf{1} \\
&= \mathbf{p}_1' \, \mathbf{Z} \, \mathbf{p}_2 \quad , \; \mathbf{Z} \; = \; \sum_{i=0}^{\infty} \mathbf{P}_1^i \, \mathbf{1} \, \mathbf{1}' \, (\mathbf{P}_2')^i
\end{aligned}
$$

$\mathbf{Z}$ solves matrix equation

$$\mathbf{Z} = \mathbf{1} \, \mathbf{1}' + \mathbf{P}_1 \, \mathbf{Z} \, \mathbf{P}_2'$$

or equivalent Sylvester matrix equation

$$
\begin{aligned}
&(\mathbf{I} - \mathbf{P}_1)^{-1} \, \mathbf{Z} + \mathbf{Z} \, \mathbf{P}_2' \, (\mathbf{I} - \mathbf{P}_2')^{-1} \\
&\quad = \; (\mathbf{I} - \mathbf{P}_1)^{-1} \, \mathbf{1} \, \mathbf{1}' \, (\mathbf{I} - \mathbf{P}_2')^{-1} \, .
\end{aligned}
$$

(Morais and Pacheo 1999: truncating)

Comparison Study

$$
\begin{aligned}
\mu_0 &= 0\,, \\
q &= 1\,, \\
\rightarrow \qquad X_{i,j} &= \Delta\, Y_{i,j} + a\,\sqrt{\gamma_0}\,, \\
Y_{i,j} &= \alpha\, Y_{i,j-1} + \varepsilon_{i,j}\,. \\
a &\in \{0, .25, \ldots, 1.5, 2\}\,, \\
\Delta &\in \{1, 1.1, \ldots, 1.5, 1.75, 2\}\,. \\
E_0(\tau) &= 500\,. \\
\lambda &\in \{.05, .1, .25, .5, 1.\}\,. \\
(n, \alpha) &= (5, .3), (10, .5), (10, -.5)\,.
\end{aligned}
$$

Monte Carlo with 10^6 repetitions.

Optimal (λ_1, λ_2) and related ARLs

$\alpha = 5$, $n = 5$

Modified chart

Δ	a 0.00	0.25	0.50	0.75	...	2.00
1.00	**500**	41.59 *(.05,.5)*	14.21 *(.1,.05)*	7.67 *(.25,.05)*	...	1.68 *(1.,.05)*
1.10	64.00 *(1.,.05)*	31.81 *(.05,.05)*	13.42 *(.1,.1)*	7.38 *(.25,.25)*	...	1.70 *(1.,.05)*
1.20	23.59 *(1.,.05)*	19.63 *(.1,.05)*	11.66 *(.1,.1)*	6.89 *(.25,.25)*	...	1.71 *(.5,1.)*
⋮						
2.00	2.48 *(1.,.25)*	2.44 *(1.,.25)*	2.37 *(1.,.25)*	2.25 *(1.,.25)*	...	1.47 *(1.,.5)*

Residual chart

Δ	a 0.00	0.25	0.50	0.75	...	2.00
1.00	**500**	41.31 *(.05,1.)*	14.17 *(.1,1.)*	7.68 *(.25,1.)*	...	1.69 *(1.,1.)*
1.10	59.40 *(1.,.05)*	31.08 *(.05,.05)*	13.32 *(.1,.25)*	7.36 *(.25,.25)*	...	1.70 *(.5,1.)*
1.20	21.94 *(1.,.05)*	18.70 *(.1,.05)*	11.37 *(.1,.1)*	6.78 *(.25,.25)*	...	1.71 *(.5,1.)*
⋮						
2.00	2.30 *(1.,.5)*	2.27 *(1.,.5)*	2.21 *(1.,.5)*	2.09 *(1.,.5)*	...	1.42 *(1.,.5)*

Conclusions

- pure shifts: both schemes act similarly,

  $a \uparrow \Rightarrow$ optimal $\lambda_1 \uparrow$

  $\Delta = 2 \rightarrow$ overall $\lambda_1 = .25/.5$,
  $a = 0 \rightarrow$ overall $\lambda_1 = 1$

- $\Delta > 1$: residual chart performs better

- modified chart:

  1. more attractive for practitioner,
  2. extremely time–consuming determination of critical values,
  3. worse performance.

- residual chart:

  1. more artificial appearance,
  2. quick determination of critical values,
  3. better performance.

- as usual, EWMA is better for small changes than Shewhart ($\lambda = 1$) chart.

References

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Example

Shewhart (1931): 204 measurements of electrical resistance (in Megohms)

51 (rational) subgroups with size 4

$\bar{x} = 4498$, $s^2 = 215\,000$

Wieringa (1999): $\hat{\alpha} = 0.55$

204 m. as **prerun**, daily sampling scheme of size $n = 4$

independence between days,
$AR(1)$ with $\alpha = 0.55$ within day

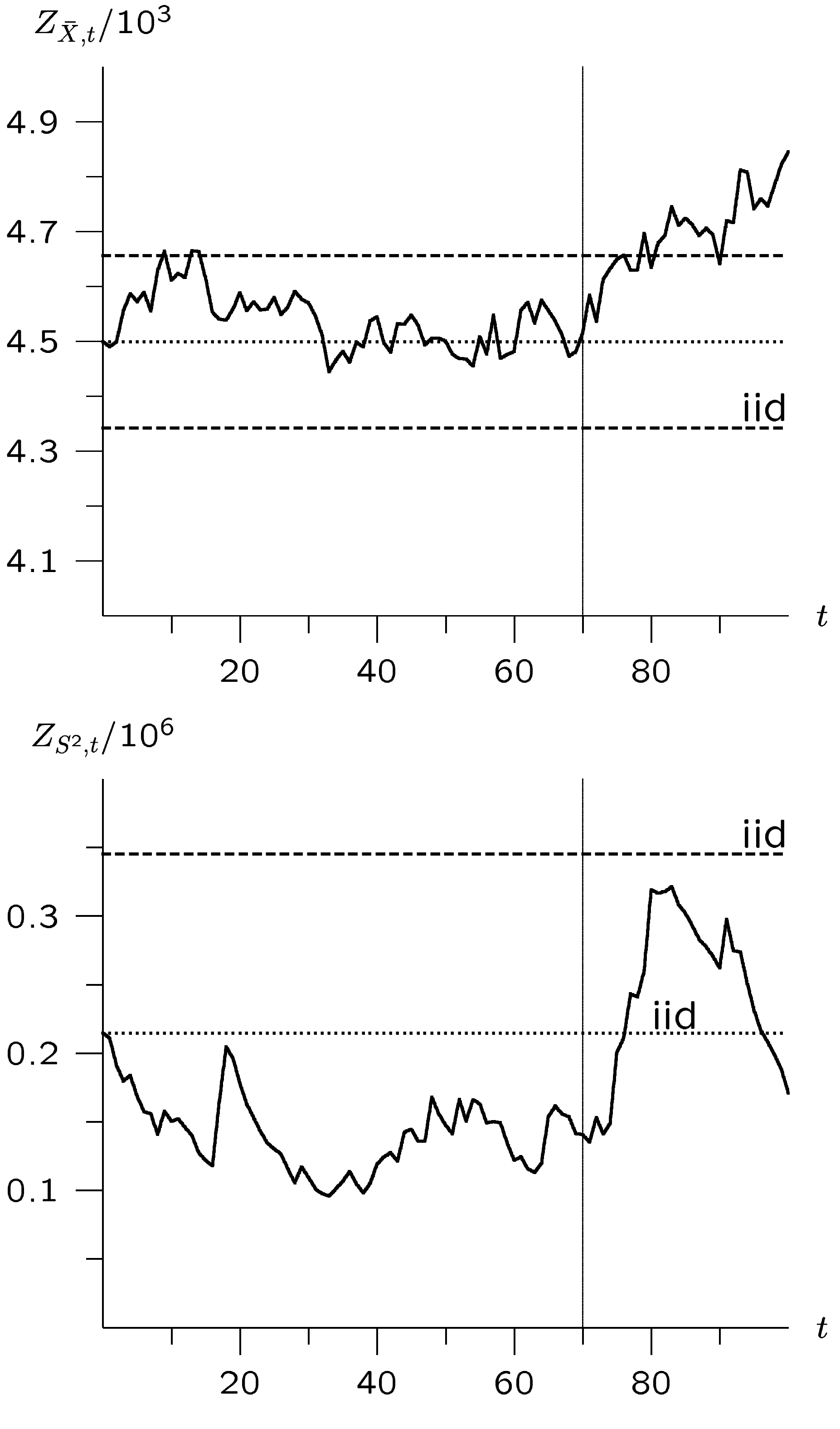
assuming iid and in–control ARL 370, $\lambda_1 = \lambda_2 = 0.1$
$\rightsquigarrow x_{iid} = 2.9521, s_{iid} = 3.2410$

with ARLs

		a	
		0	1
Δ	1	370.0	11.06
	1.3	14.85	8.41

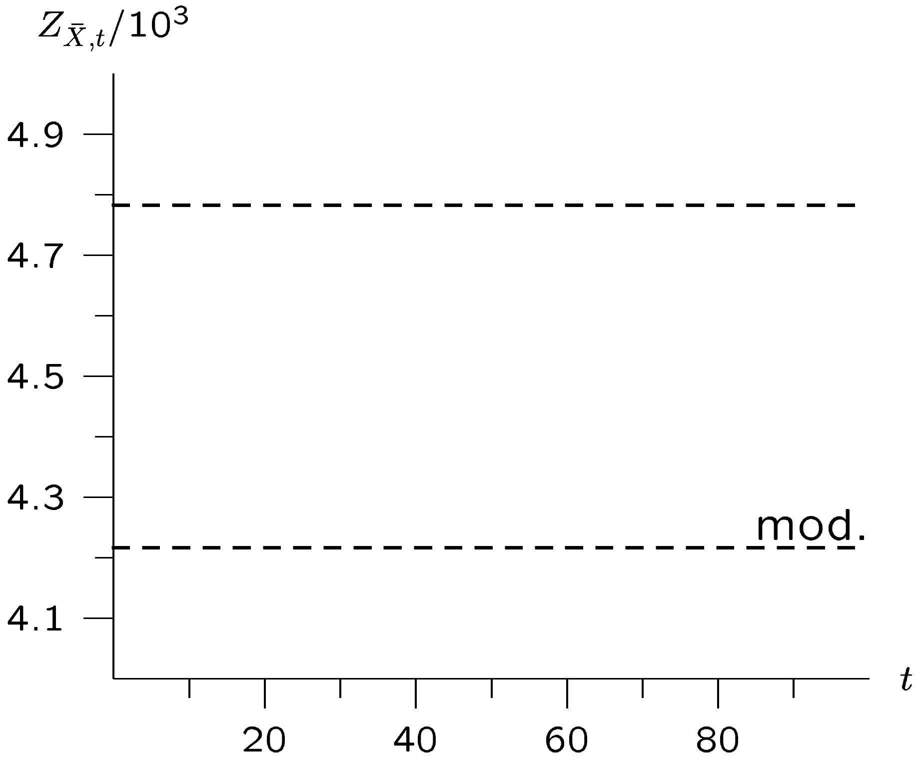
EWMA scheme

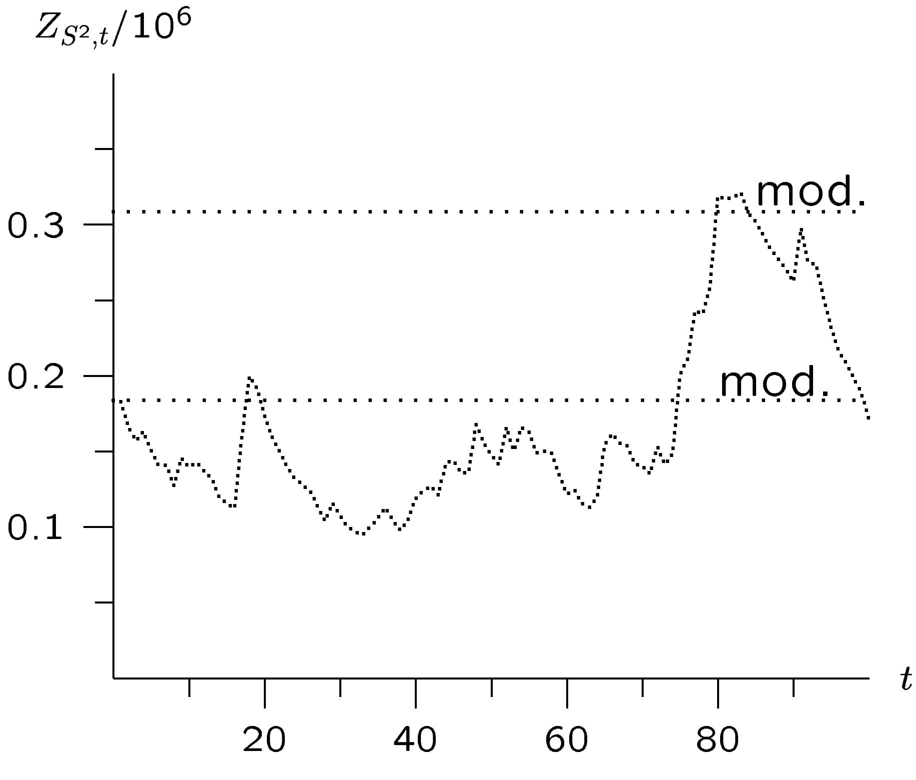
iid



EWMA scheme

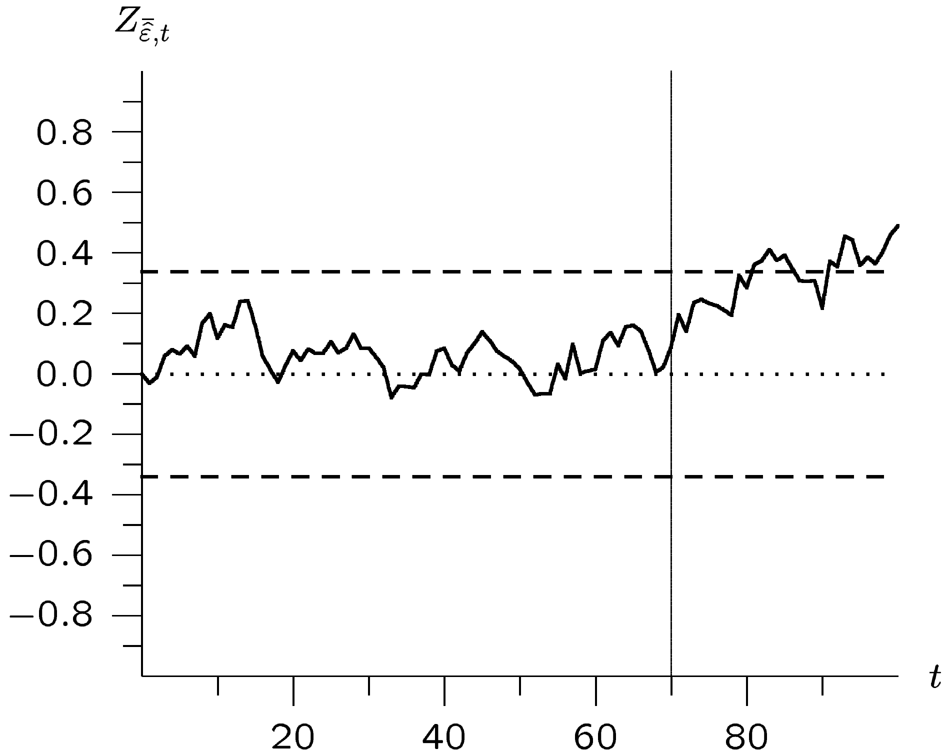
modified

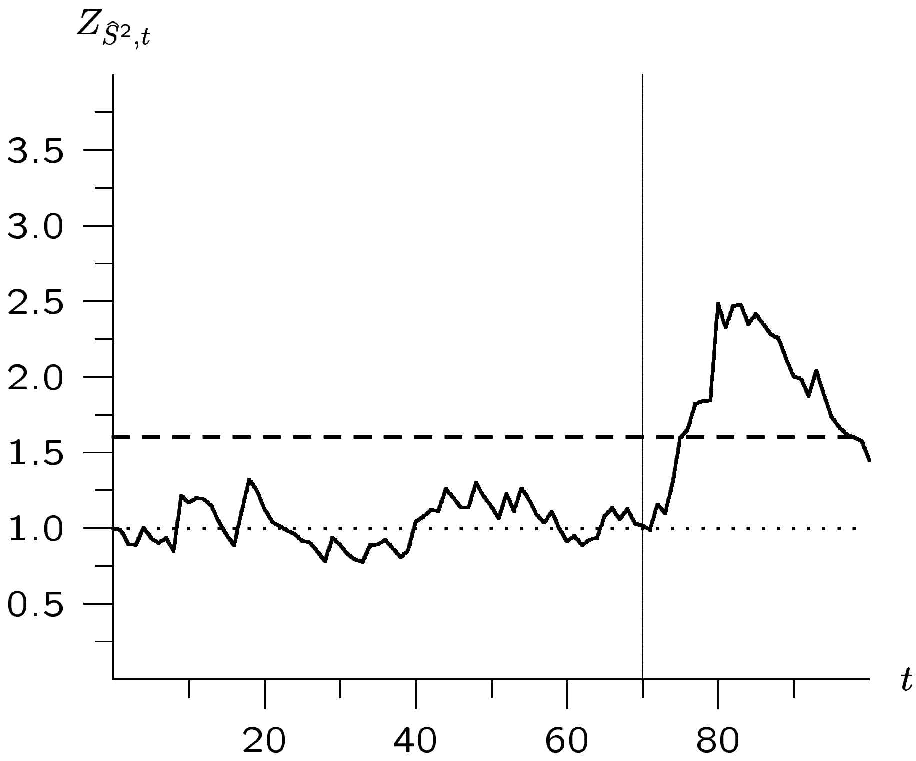




EWMA scheme

residual





related ARLs

ideal

		a = 0	a = 1
Δ	1	370.0	11.06
	1.3	14.85	8.41

actual

		a = 0	a = 1
Δ	1	71.22	10.60
	1.3	27.17	9.89

modified

		a = 0	a = 1
Δ	1	375.7	20.64
	1.3	17.85	11.98

residual

		a = 0	a = 1
Δ	1	370.3	21.49
	1.3	14.84	10.91

MC with 10^6 repetitions